

Superfunctions and the algebra of subspace collections and their association with rational functions of several complex variables

Graeme W. Milton

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

Abstract

A natural connection between rational functions of several real or complex variables, and subspace collections is explored. A new class of function, superfunctions, are introduced which are the counterpart to functions at the level of subspace collections. Operations on subspace collections are found to correspond to various operations on rational functions, such as addition, multiplication and substitution. It is established that every rational matrix valued function which is homogeneous of degree 1 can be generated from an appropriate, but not necessarily unique, subspace collection: the mapping from subspace collections to rational functions is onto, but not one to one. For some applications superfunctions may be more important than functions, as they incorporate more information about the physical problem, yet can be manipulated in much the same way as functions. Previously subspace collections had been introduced when there was an inner product on the vector (or Hilbert) space, and appropriate subspaces were mutually orthogonal. In that setting certain normalization and reduction operations on subspace collections led to a continued fraction expansion of the associated function, which allowed one to bound the function in terms of a set of weight matrices and normalization matrices that are derived from series expansions. Here we also initiate the theory of normalization and reduction operations, appropriate when there is no inner product on the space.

1 Introduction

This Chapter 7 of the book "Extending the Theory of Composites to other Areas of Science", edited by Graeme W. Milton, is concerned with developing the theory of subspace collections, particularly nonorthogonal subspace collections. Subspace collections have a rich algebraic structure, and a close connection with rational functions of several real or complex variables. Here we are interested in three types of subspace collections: first, finite dimensional vector spaces \mathcal{H} that have the decomposition

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (1.1)$$

which we call a $Z(n)$ subspace collection; second finite dimensional vector spaces \mathcal{K} (over the real or complex numbers) that have the decomposition

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (1.2)$$

which we call a $Y(n)$ subspace collection, where the \mathcal{E} and \mathcal{J} entering (1.2) are not to be confused with the subspaces \mathcal{E} and \mathcal{J} entering (1.1); and third finite dimensional vector spaces \mathcal{K} (over the real or complex numbers) that have the decomposition

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V}^I \oplus \mathcal{V}^O \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (1.3)$$

which we call a superfunction $F^s(n)$. In a superfunction the space \mathcal{V}^I and the space \mathcal{V}^O are called the input and output subspaces respectively, and they have the same dimension. For superfunctions we require the technical condition that for any choice of vectors $\mathbf{E}^I, \mathbf{J}^I \in \mathcal{V}^I$ and $\mathbf{E}^O, \mathbf{J}^O \in \mathcal{V}^O$ there exist vectors $\mathbf{E} \in \mathcal{E}$ and $\mathbf{J} \in \mathcal{J}$ such that

$$\mathbf{E}^I = \Pi^I \mathbf{E}, \quad \mathbf{J}^I = \Pi^I \mathbf{J}, \quad \mathbf{E}^O = \Pi^O \mathbf{E}, \quad \mathbf{J}^O = \Pi^O \mathbf{J}. \quad (1.4)$$

As we will see there is a very close direct connection between a superfunction $F^s(n)$ and a $Y(n)$ subspace collections, and also many connections between them and $Z(n)$ subspace collections. All are intertwined and that is the beauty of the theory. $Z(3)$ and $Y(2)$ subspace collections, and superfunctions $F^s(1)$ can be visualized in 3-dimensional space, and examples of these are given Figure 1.

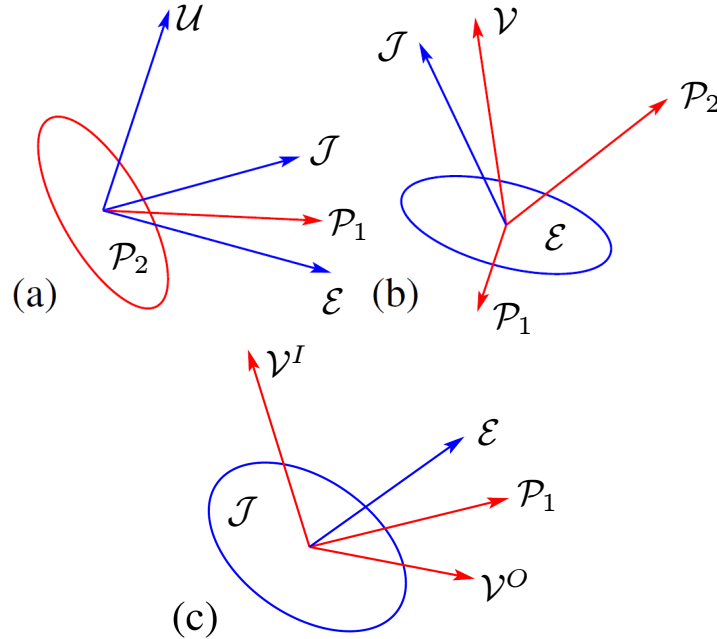


Figure 1: Shown in (a) is an example of a $Z(3)$ subspace collection, in (b) a $Y(2)$ subspace collection, and in (c) a superfunction $F^s(1)$. The rays denote one-dimensional subspaces: they should really be drawn as lines, but for clarity they are drawn as rays and should be extended in the opposite direction as the ray. The circles, which look like ellipses as they are tilted, represent two-dimensional subspaces.

One reason $Y(n)$ subspace collections, $Z(n)$ subspaces collections, and superfunctions $F^s(n)$ are important is because they arise in many physical problems. For examples in network theory and in the theory of the effective moduli of composite materials, see the review in Chapter 2 of this book (Milton 2016) and Milton (2002). There are also many other physical problems where subspace collections arise as is apparent in Chapters 1,3,8,9, 12, 13, and 14 of this book (Milton 2016). In physics applications the subspaces are usually orthogonal with respect to some inner product on the space \mathcal{H} or \mathcal{K} but as this chapter shows the theory of them can be developed without reference to an inner product. This generalization is important to make contact between general rational functions of complex variables, thus extending the notion of a function: hence the name superfunction. The generalization is also important for applications, such as speeding up numerical methods for calculating the fields that solve the problem: we will see an example of this in the next chapter.

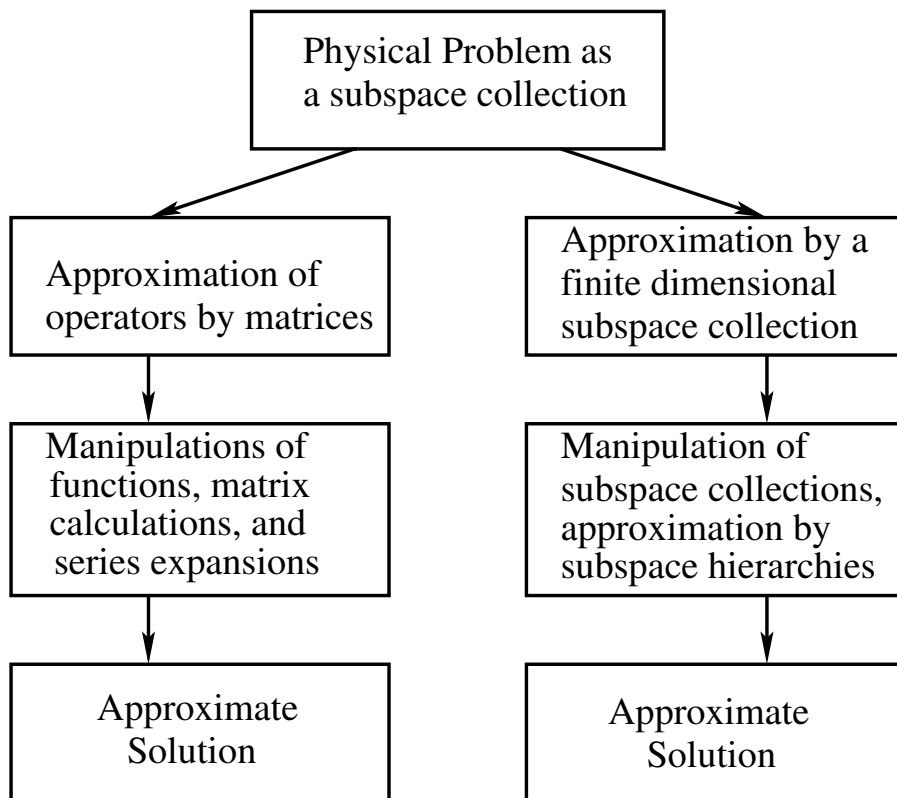


Figure 2: Two routes to solving a physical problem formulated in terms of subspace collections. It is suggested that the route on the right may result in a better approximation as more information is kept.

It may very well be the case that superfunctions become more important than functions in some applications, as suggested by the flow chart of Figure 2. The reason is that when one extracts the function from a superfunction, which we will see how to do shortly, one generally loses information that is contained in a superfunction. For example, in the context of physical problems where there is an inner product on the space this information may come in the form of a series expansion for the fields up to a given order, and from this series expansion one can extract the “weight matrices” and “normal-

ization matrices”, introduced by Milton and Golden (1985) and Milton (1987a, 1987b). These matrices basically encode the information about the “angles” between the various subspaces (when there is an inner product). One can then develop a continued fraction expansion for the function associated with the superfunction, with the normalization factors and weight matrices that enter it at each level having the property that they are positive semidefinite, with the weight matrices summing to one. Truncating the continued fraction gives approximations to the function, that are similar in some respects the diagonal Padé approximants, and in fact give bounds on the function if the truncation is done appropriately. The information contained in the weight matrices and normalization matrices, cannot in general be recovered (at least when $n \geq 4$) from the series expansion of the associated function. (Although one can potentially determine these matrices from the series expansion of the functions associated with coupled field problems, as shown in Chapter 9 of this book (Milton 2016)). This theory was established by Milton and Golden (1985) and Milton (1987a, 1987b, 1991). (see also Chapters 19, 20 and 29 in Milton (2002)) for the case of $Z(n)$ subspace collections, for any integer $n \geq 1$. In this paper develop the basic theory of subspace collections in the case where there is no inner product on the vector space \mathcal{H} or \mathcal{K} . We also make the first steps towards generating continued fraction expansions in the case where there is no inner product on the vector space \mathcal{H} or \mathcal{K} .

Let us first suppose \mathcal{V} and \mathcal{U} are one-dimensional. We will see that there are generally homogeneous (of degree 1) rational functions $Y(z_1, z_2, \dots, z_n)$ and $Z(z_1, z_2, \dots, z_n)$ (over the real or complex numbers) of degree 1 that are associated respectively with these $Y(n)$ and $Z(n)$ subspace collections, where $Z(z_1, z_2, \dots, z_n)$ satisfies the additional constraint that $Z(1, 1, \dots, 1) = 1$. Conversely, we will see that given any rational functions $Y(z_1, z_2, \dots, z_n)$ and $Z(z_1, z_2, \dots, z_n)$ with these properties, then there exists at least one subspace collection realizing these functions as its associated function. There are also operations on these subspace collections that correspond to operations on the associated function, such as substitution.

For superfunctions the simplest case is when the input and output spaces \mathcal{V}^I and \mathcal{V}^O are one-dimensional. Then with a specific basis for \mathcal{V}^I and \mathcal{V}^O the corresponding function $\mathbf{F}(z_1, z_2, \dots, z_n)$ is 2 by 2 matrix valued with the elements $F_{11}(z_1, z_2, \dots, z_n)$ and $F_{22}(z_1, z_2, \dots, z_n)$ being homogeneous of degree zero, the element $F_{12}(z_1, z_2, \dots, z_n)$ being homogeneous of degree minus 1, and $F_{21}(z_1, z_2, \dots, z_n)$ being homogeneous of degree 1. There are operations on superfunctions that correspond to addition, multiplication and forming an inverse (and hence division) of the associated functions. So superfunctions form an algebra. Also one can do substitutions at the level of subspace collections. Actually the operation of addition of superfunctions are naturally done with the associated Y -problem, although one could equally do them with the associated inverse Y -problem (where the spaces \mathcal{E} and \mathcal{J} are interchanged). Thus there is an inherent ambiguity of how one wants to define addition of superfunctions. The definitions of addition, multiplication. and substitution of subspace collections may seem a little complicated and abstract, yet they are the exact counterpart of similar operations one may do on multiterminal electrical networks, and they do produce the corresponding action on the associated functions. (In fact it was thinking about electrical circuits which guided the construction of these operations in a more general setting).

When \mathcal{V} and \mathcal{U} have dimension greater than 1, then $Y(z_1, z_2, \dots, z_n)$ and $Z(z_1, z_2, \dots, z_n)$

get replaced by linear operator valued functions $\mathbf{Y}(z_1, z_2, \dots, z_n)$ and $\mathbf{Z}(z_1, z_2, \dots, z_n)$ which map \mathcal{V} to \mathcal{V} and \mathcal{U} to \mathcal{U} respectively. Similarly, the function $\mathbf{F}(z_1, z_2, \dots, z_n)$ should really be thought of as a linear operator mapping \mathcal{V}^I to \mathcal{V}^O

The original motivation for studying subspace collections, and their associated functions, arose from the study of the effective conductivity tensor \mathbf{Z} of periodic composite materials. For a composite with n isotropic phases, with scalar conductivities z_1, z_2, \dots, z_n , the effective conductivity tensor was found to be a homogeneous (of degree 1) analytic function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ of the component conductivities with positive definite imaginary part when the component conductivities have positive imaginary part [Bergman 1978; Milton 1979, 1981a, Golden and Papanicolaou 1983] (see also Chapter 18 of Milton (2002)). It was also recognized (Milton 1987a, 1990) that the problem of determining the effective conductivity function could be formulated in terms of three mutually orthogonal spaces in the Hilbert space \mathcal{H} of square integrable functions: namely the space \mathcal{U} of constant fields, the space \mathcal{E} of periodic square integrable electric fields (having zero curl), and the space \mathcal{J} of square integrable current fields (having zero divergence), and if the composite had n isotropic phases, with conductivities z_1, z_2, \dots, z_n , then it was also natural to decompose \mathcal{H} into the direct sum of n mutually orthogonal subspaces $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ where \mathcal{P}_i consists of those square integrable fields which are nonzero only within component i . This formulation, in terms of a $Z(n)$ subspace collection, evolved out of earlier Hilbert space formulations of the problem (Fokin 1982; Kohler and Papanicolaou 1982; Papanicolaou and Varadhan 1982; Golden and Papanicolaou 1983; Kantor and Bergman 1984; Dell’Antonio, Figari, and Orlandi 1986) and can easily be extended to the elastic, thermoelastic, piezoelectric, and poroelastic equations of multiphase and polycrystalline materials (see, for example, Chapter 12 in Milton (2002)). The formulation has proved to be particularly important in the theory of exact relations of composite materials (Grabovsky 1998; Grabovsky and Sage 1998; Grabovsky and Milton 1998; Grabovsky, Milton, and Sage 2000; Grabovsky 2004) (see also Chapter 17 in Milton (2002)) where one seeks microstructure independent relations satisfied by effective tensors. For two-dimensional polycrystals a complete correspondence was established between subspace collections and a representative class of multiple rank laminate polycrystal geometries (Clark and Milton 1994), thus showing that the subspace collection of any two-dimensional polycrystal, with any configuration of crystal grains, could be approximated arbitrarily closely by the subspace collection of one of these multiple rank laminate polycrystal geometries.

Curiously the connection between $Z(n)$ subspace collections and the effective conductivity allowed the effective conductivity function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ to be expanded as a new type of continued fraction, involving matrices of increasing dimension as one proceeds down the continued fraction when $n > 2$ (Milton 1987a, 1987b, 1991; see also Chapters 19, 20 and 29 in Milton 2002). The coefficients in the weight and normalization matrices entering the continued fraction can be expressed in terms of inner products between fields that enter the series expansion of the solution field in a nearly homogeneous medium (with all the conductivities z_1, z_2, \dots, z_n being close to one another). One application of the continued fraction expansion has been to obtain bounds on the diagonal elements of the complex effective conductivity tensor of a three phase conducting composite, with complex conductivities z_1, z_2 and z_3 , that were tighter than bounds obtained by any other method (see figure 4 in Milton 1987b). This procedure essentially extended to

multivariate functions the procedure, using successive fractional linear transformations, that was used to obtain bounds (Baker, Jr. 1969) on the values in the complex plane that Stieltjes functions! can take when a finite number of Taylor series coefficients are known (see also Golden and Papanicolaou 1983; Bergman 1986) where essentially the same transformation is used to derive bounds on the complex dielectric constant of two component media using series expansion coefficients, as noted in the appendix in Milton 1986, and see Milton 1981b, where the same set of bounds is derived using a different procedure, namely the method of variation of poles and zeros.)

In the case $n = 2$ the continued fraction reduces to a usual continued fraction expansion, like those continued fractions associated with Padé approximants (see Chapter 4 of Part I of Baker, Jr. and Graves-Morris 1981). $Y(n)$ subspace collections enter, for example, if one eliminates from the Hilbert space the constant fields and then reformulates the conductivity equations in terms of the remaining fields: the driving fields are then fields which are constant in each phase, but have zero average value (see Chapter 19 in Milton 2002 and references therein). The interrelationship between $Z(n)$ subspace collections and $Y(n)$ subspace collections is what gives rise to these novel continued fractions.

Finite dimensional $Z(n)$ and $Y(n)$ subspace collections also arise naturally in the study of the effective resistance of electrical circuits constructed from n types of resistors having conductances z_1, z_2, \dots, z_n (see Chapter 20 in Milton 2002). This is not surprising as periodic resistor networks can be seen as discrete approximations to conducting composite materials (see, for example, Milton 1981a and Figure 8.5(a) in this book Milton 2016). Figure 3 illustrates a discrete network of impedances, and gives an indication of the physical meaning of the $Z(n)$ and $Y(n)$ subspace collections in this context.

In this figure, the vector space \mathcal{H} is 6-dimensional, and is the direct sum of the two-dimensional space \mathcal{P}_1 consisting of fields that are nonzero only along the resistors $c_1 z_1$ and $c_3 z_1$; the two-dimensional space \mathcal{P}_2 consisting of fields that are nonzero only along the resistors $c_2 z_2$ and $c_5 z_2$; and the one-dimensional space \mathcal{P}_3 consisting of fields that are nonzero only along the resistor $c_4 z_3$. The response of the network, when one terminal is grounded (with zero voltage) is a 3×3 matrix. When it acts on the vector, having as elements the voltages at the three remaining terminals, it gives the three currents flowing through these terminals. The 3×3 matrix valued function $\mathbf{Z}(z_1, z_2, z_3)$ gives the matrix valued response relative to the response when $z_1 = z_2 = z_3 = 1$. Now, let us imagine all the resistors, or impedances, in (a) are on one side of the circuit board, with the terminals being conducting posts that penetrate the board. On the other side of the board these posts are connected to a tree-like graph of batteries (or alternating current sources if the fields vary sinusoidally in time) shown in (b). The three fields in these batteries constitute the space \mathcal{V} . The $Y(3)$ subspace collection contains fields on both sides of the board, in $\mathcal{K} = \mathcal{H} \oplus \mathcal{V}$. The associated 3×3 matrix valued Y -function $\mathbf{Y}(z_1, z_2, z_3)$ gives the current going through the three batteries, in response to the voltages across them. Note that $\mathbf{Y}(z_1, z_2, z_3)$ is not diagonal: a voltage across one battery, sends current through the other two batteries, even when they have zero voltage across them.

Superfunctions are a natural generalization of multiport electrical circuits with input ports and output ports, as illustrated in Figure 4. The function \mathbf{F} gives the currents and potential drops across the output batteries/resistors that are generated in response to currents and potential drops across the input batteries. Note that the networks associated with superfunctions automatically satisfy the “port condition” that the net flow of current

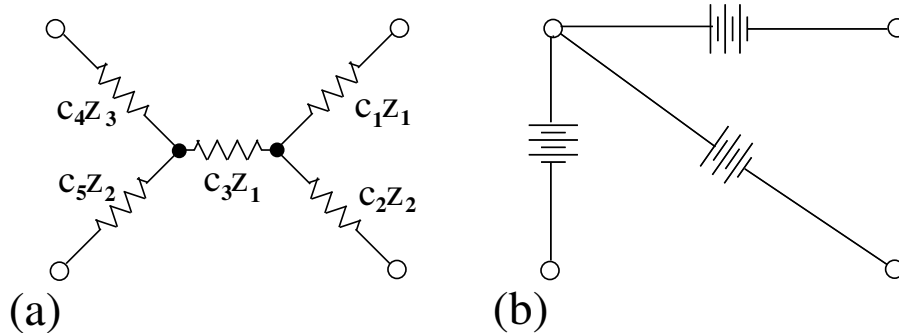


Figure 3: Shown in (a) is a 4 terminal electrical network, which is representative of a $Z(3)$ subspace collection. Here the c_i are real positive scaling constants: the conductance of each element is $c_j z_k$ where z_k is real or complex (when z is complex we should refer to $c_j z_k$ as an admittance rather than as a conductance). Complex values of z are appropriate when the applied potentials vary sinusoidally with time, and some of the impedance elements are capacitors or inductors. Figure (b) shows the batteries on the back side of the circuit board, representing the space \mathcal{V} , which combined with the resistors on the front side is representative of a $Y(3)$ subspace collection. The Y -function $\mathbf{Y}(z_1, z_2, z_3)$ gives the current going through the three batteries, in response to the voltages across them.

from the input terminals to the output terminals is zero.

In this chapter we show that the connection between finite dimensional $Z(n)$ and $Y(n)$ subspace collections and homogeneous (degree 1) operator valued rational functions $\mathbf{Y}(z_1, z_2, \dots, z_n)$ and $\mathbf{Z}(z_1, z_2, \dots, z_n)$ persists even when the subspaces in each decomposition are not necessarily mutually orthogonal, and indeed even in the absence of an inner product (on the space \mathcal{H} or \mathcal{K}). The results developed in (Milton, 1987a, 1987b, 1991 and in Chapters 19, 20 and 29 of Milton, 2002) are extended to the case where there is no inner product. Accordingly some steps in the analysis, and some assumptions, need to be revised. In this more general setting we can generate, from an appropriate $Z(n)$ subspace collection, any desired scalar valued rational function $Z(z_1, z_2, \dots, z_n)$ satisfying the homogeneity property $Z(1, 1, \dots, 1) = 1$.

It is to be emphasized that subspace collections, with the associated rules for addition, multiplication and substitution, are algebraic objects in their own right: there is no need to think of the associated analytic functions (that are in general operator valued), except that the correspondence makes it easier to think about subspace collections. The resistor network examples of $Y(n)$ subspace collections made it possible for me to see how the operations of addition, multiplication and substitution of subspace collections should be defined in the general case.

My belief is that the geometrical structure of subspace collections (and in particular superfunctions) will be reflected in the algebraic geometrical structure of their associated rational functions. If this is the case, understanding the topological features of subspace collections might shed light on the geometrical features of algebraic varieties. While this paper does not directly address this issue, it sheds the first light on the relation between finite dimensional subspace collections and rational functions of several complex variables, in the case where the subspaces are not mutually orthogonal, and it introduces

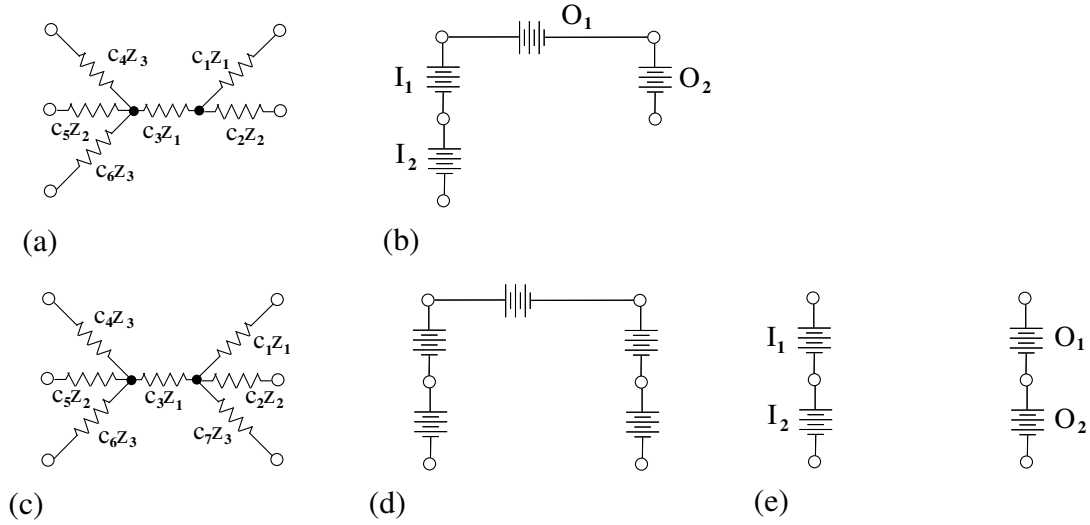


Figure 4: Shown in (a) is a 5 terminal electrical network, which is representative of a $Z(3)$ subspace collection. Here the c_i are real positive scaling constants: the admittance of each element is $c_j z_k$ where z_k is real or complex. Figure (b) shows the batteries on the back side of the circuit board, representing the space \mathcal{V} , which is divided into the input space \mathcal{V}^I , consisting of those vectors in \mathcal{K} that are nonzero only in the batteries I_1 and I_2 and the output space \mathcal{V}^O , consisting of those vectors in \mathcal{K} that are nonzero only in the batteries/resistors O_1 and O_2 . Figure (c) shows a 6 terminal electrical network, and the naturally associated subspace \mathcal{V} represented by the batteries in Figure (d). To convert this to a problem where the dimension of \mathcal{V} is even we remove the battery at the top, and accordingly reduce the dimension of both \mathcal{V} and \mathcal{J} by one. Figure (e) shows the input space \mathcal{V}^I , consisting of those vectors that are nonzero only in the batteries I_1 and I_2 and the output space \mathcal{V}^O , consisting of those vectors in \mathcal{K} that are nonzero only in the batteries/resistors O_1 and O_2 .

superfunctions. The functions derived from superfunctions are well studied and have widespread applications in signal processing, control theory, network synthesis and design, and in optics, acoustics and elastodynamics (usually in layered media), where they are called a variety of names including transfer matrices, transmission matrices, transfer functions, system functions, and network functions. In these contexts it is the function that is studied, but people do not think of the superfunction. I thank Aaron Welters and Mihai Putinar for drawing my attention to the connection between transfer functions and response functions (such the effective conductivity tensor of composites).

We remark that for $Z(3)$ orthogonal subspace collections, with \mathcal{U} being one-dimensional, it is still an open and intriguing question as to whether there could be a one-to-one correspondence between them (assuming they are pruned as described in Section 15 and modulo trivial equivalences between subspace collections) and scalar functions $Z(z_1, z_2, z_3)$ satisfying the homogeneity, Herglotz and normalization properties. The Z -problem described the next section provides a nonlinear map from the $Z(3)$ orthogonal subspace collection to an associated scalar function $Z(z_1, z_2, z_3)$ satisfying the homogeneity, Herglotz and normalization properties, but the question is whether one can uniquely recover the pruned subspace collection, modulo trivial equivalences, given only the function

$Z(z_1, z_2, z_3)$? The intriguing counting argument given in Section 29.2 of Milton (2002) suggests the possibility of a one-to-one correspondence. There is a similar counting argument for nonorthogonal subspace collections given in Section 18, but in this case we will see in an explicit example that a one-to-one correspondence does not hold.

2 Subspace collections and their associated functions

Let \mathcal{K} be a vector space which has a decomposition into two different direct sums of subspaces

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{H}, \quad (2.1)$$

where \mathcal{H} itself is a direct sum of n subspaces

$$\mathcal{H} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n. \quad (2.2)$$

Any vector $\mathbf{K} \in \mathcal{K}$ has a unique decomposition into component vectors,

$$\mathbf{K} = \mathbf{E} + \mathbf{J} = \mathbf{v} + \mathbf{H}, \quad \mathbf{H} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n, \quad (2.3)$$

each in the associated subspaces:

$$\mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{v} \in \mathcal{V}, \quad \mathbf{H} \in \mathcal{H}, \quad \mathbf{P}_i \in \mathcal{P}_i \text{ for } i = 1, 2, \dots, n. \quad (2.4)$$

This decomposition serves to define projection operators Γ_1 and Γ_2 onto \mathcal{E} and \mathcal{J} , projection operators Π_1 and Π_2 onto \mathcal{V} and \mathcal{H} , and projection operators Λ_i onto the subspaces \mathcal{P}_i . By definition we have

$$\mathbf{E} = \Gamma_1 \mathbf{K}, \quad \mathbf{J} = \Gamma_2 \mathbf{K}, \quad \mathbf{v} = \Pi_1 \mathbf{K}, \quad \mathbf{H} = \Pi_2 \mathbf{K}, \quad \mathbf{P}_i = \Lambda_i \mathbf{K}. \quad (2.5)$$

Associated with this subspace collection is an linear operator valued function $\mathbf{Y}(z_1, z_2, \dots, z_n)$ acting on the space \mathcal{V} , which is a homogeneous function of degree 1 of the n complex variables z_1, z_2, \dots, z_n . To obtain the function we take each field $\mathbf{E}_1 \in \mathcal{V}$ and look for vectors \mathbf{J} and \mathbf{E} that solve the equations

$$\mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{J}_2 = \mathbf{L}\mathbf{E}_2, \quad \text{where } \mathbf{J}_2 = \Pi_2 \mathbf{J}, \quad \mathbf{E}_2 = \Pi_2 \mathbf{E}, \quad (2.6)$$

with $\mathbf{E}_1 = \Pi_1 \mathbf{E}$, where

$$\mathbf{L} = \sum_{i=1}^n z_i \Lambda_i. \quad (2.7)$$

We call this problem the Y -problem. The associated operator \mathbf{Y} , by definition, governs the linear relation

$$\mathbf{J}_1 = -\mathbf{Y}\mathbf{E}_1, \quad \text{where } \mathbf{J}_1 = \Pi_1 \mathbf{J}. \quad (2.8)$$

A necessary condition for \mathbf{J}_1 to be uniquely defined given \mathbf{E}_1 is that

$$\mathcal{V} \cap \mathcal{J} = 0, \quad (2.9)$$

since if \mathbf{J} and \mathbf{E} solve (2.6) so too will $\mathbf{J} + \mathbf{v}$ and \mathbf{E} , for any $\mathbf{v} \in \mathcal{V} \cap \mathcal{J}$. The inverse Y -problem is to solve (2.6) for each field $\mathbf{J}_1 = \Pi_1 \mathbf{J} \in \mathcal{V}$. A necessary condition for \mathbf{E}_1 to be uniquely defined given \mathbf{J}_1 is that

$$\mathcal{V} \cap \mathcal{E} = 0. \quad (2.10)$$

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ is a basis of \mathcal{V} , then the operator \mathbf{Y} can be represented by a matrix, the Y -matrix, also denoted by \mathbf{Y} with elements Y_{ik} such that

$$\mathbf{Y}\mathbf{v}_k = \sum_{i=1}^m Y_{ik}\mathbf{v}_i. \quad (2.11)$$

If m is even and \mathcal{V} has the decomposition

$$\mathcal{V} = \mathcal{V}^I \oplus \mathcal{V}^O, \quad (2.12)$$

where \mathcal{V}^I and \mathcal{V}^O have the same dimension ($m/2$) then we have a superfunction F^s . The superfunction is the collection of subspaces and there is a function \mathbf{F} associated with it. The fields \mathbf{E}_1 and \mathbf{J}_1 have the unique decomposition

$$\mathbf{E}_1 = \mathbf{E}^I + \mathbf{E}^O, \quad \mathbf{J}_1 = \mathbf{J}^I + \mathbf{J}^O, \quad (2.13)$$

with

$$\mathbf{E}^I, \mathbf{J}^I \in \mathcal{V}^I, \quad \mathbf{E}^O, \mathbf{J}^O \in \mathcal{V}^O, \quad (2.14)$$

where the superscripts I and O refer to input and output respectively. We write

$$\mathbf{E}^I = \mathbf{\Pi}^I \mathbf{E}_1, \quad \mathbf{E}^O = \mathbf{\Pi}^O \mathbf{E}_1, \quad \mathbf{J}^I = \mathbf{\Pi}^I \mathbf{J}_1, \quad \mathbf{J}^O = \mathbf{\Pi}^O \mathbf{J}_1, \quad (2.15)$$

which defines the projections $\mathbf{\Pi}^I$ and $\mathbf{\Pi}^O$ onto the input and output spaces. Now the relation (2.8) can be written as

$$\begin{pmatrix} \mathbf{J}^I \\ \mathbf{J}^O \end{pmatrix} = \begin{pmatrix} \mathbf{Y}^{II} & \mathbf{Y}^{IO} \\ \mathbf{Y}^{OI} & \mathbf{Y}^{OO} \end{pmatrix} \begin{pmatrix} \mathbf{E}^I \\ \mathbf{E}^O \end{pmatrix}, \quad (2.16)$$

and manipulated into the form

$$\begin{pmatrix} \mathbf{E}^O \\ \mathbf{J}^O \end{pmatrix} = \mathbf{F} \begin{pmatrix} \mathbf{E}^I \\ \mathbf{J}^I \end{pmatrix}, \quad (2.17)$$

which defines the linear operator valued function

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}^{EE} & \mathbf{F}^{EJ} \\ \mathbf{F}^{JE} & \mathbf{F}^{JJ} \end{pmatrix} = \begin{pmatrix} -(\mathbf{Y}^{IO})^{-1}\mathbf{Y}^{II} & -(\mathbf{Y}^{IO})^{-1} \\ [\mathbf{Y}^{OO}(\mathbf{Y}^{IO})^{-1}\mathbf{Y}^{II} - \mathbf{Y}^{OI}] & \mathbf{Y}^{OO}(\mathbf{Y}^{IO})^{-1} \end{pmatrix}, \quad (2.18)$$

which, provided the operator \mathbf{Y}^{IO} is nonsingular, is the function associated with the superfunction. This relation can be inverted to yield \mathbf{Y} in terms of \mathbf{F} ,

$$\mathbf{Y} = \begin{pmatrix} (\mathbf{F}^{EJ})^{-1}\mathbf{F}^{EE} & -(\mathbf{F}^{EJ})^{-1} \\ [\mathbf{F}^{JJ}(\mathbf{F}^{EJ})^{-1}\mathbf{F}^{EE} - \mathbf{F}^{JE}] & -\mathbf{F}^{JJ}(\mathbf{F}^{EJ})^{-1} \end{pmatrix}, \quad (2.19)$$

provided the operator \mathbf{F}^{EJ} can be inverted. The superfunction problem is for given input fields \mathbf{E}^I and \mathbf{J}^I to find fields \mathbf{E} and \mathbf{J} that solve the Y -problem (2.6) and (2.7), with $\mathbf{\Pi}^I \mathbf{E} = \mathbf{E}^I$ and $\mathbf{\Pi}^I \mathbf{J} = \mathbf{J}^I$. It may happen that the superfunction problem has a solution when the Y -problem does not (this happens when \mathbf{F}^{EJ} is singular), and conversely the Y -problem may have a solution when the superfunction problem does not (this happens when \mathbf{Y}^{IO} is singular).

Another association between subspace collections and functions comes if a vector space \mathcal{H} has the decomposition

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (2.20)$$

where \mathcal{E} and \mathcal{J} are not to be confused with the spaces in (2.1). Any vector $\mathbf{H} \in \mathcal{H}$ has a unique decomposition into component vectors,

$$\mathbf{H} = \mathbf{u} + \mathbf{E} + \mathbf{J} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n, \quad (2.21)$$

each in the associated subspaces:

$$\mathbf{u} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{P}_i \in \mathcal{P}_i \text{ for } i = 1, 2, \dots, n. \quad (2.22)$$

This decomposition serves to define projection operators Γ_0 , Γ_1 and Γ_2 onto \mathcal{U} , \mathcal{E} and \mathcal{J} , and projection operators Λ_i onto the subspaces \mathcal{P}_i . Associated with this subspace collection is an linear operator valued function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ acting on the space \mathcal{U} , which is a homogeneous function of degree 1 of the n complex variables z_1, z_2, \dots, z_n . To obtain the function we take each vector $\mathbf{e} \in \mathcal{U}$ and look for vectors \mathbf{j} , \mathbf{J} and \mathbf{E} that solve the equations

$$\mathbf{j} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{j} + \mathbf{J} = \mathbf{L}(\mathbf{e} + \mathbf{E}), \quad \text{where } \mathbf{L} = \sum_{i=1}^n z_i \Lambda_i. \quad (2.23)$$

We call this problem the Z -problem. The associated operator \mathbf{Z} , by definition, governs the linear relation

$$\mathbf{j} = \mathbf{Z}\mathbf{e}. \quad (2.24)$$

If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is a basis of \mathcal{U} , then the operator \mathbf{Z} can be represented by a matrix, also denoted by \mathbf{Z} with elements Z_{ik} such that

$$\mathbf{Z}\mathbf{u}_k = \sum_{i=1}^m Z_{ik} \mathbf{u}_i. \quad (2.25)$$

When $z_1 = z_2 = \cdots = z_n = 1$ (2.23) has the trivial solution

$$\mathbf{j} = \mathbf{e}, \quad \mathbf{J} = \mathbf{E} = 0, \quad (2.26)$$

and so we deduce that

$$\mathbf{Z}(1, 1, \dots, 1) = \mathbf{I}. \quad (2.27)$$

The inverse Z -problem is to solve the equations (2.23) for each given vector $\mathbf{j} \in \mathcal{U}$.

3 Some simple examples

Consider a $Y(n)$ subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (3.1)$$

where $\mathcal{E}, \mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ are all one-dimensional, and \mathcal{J} is n -dimensional. Choose, as our basis for \mathcal{K} , $n+1$ vectors $\mathbf{p}_0 \in \mathcal{V}$, and $\mathbf{p}_i \in \mathcal{P}_i$, $i = 1, 2, \dots, n$. Vectors $\mathbf{E} \in \mathcal{E}$ and $\mathbf{J} \in \mathcal{J}$ can be expanded in this basis:

$$\mathbf{E} = \sum_{i=0}^n E_i \mathbf{p}_i, \quad \mathbf{J} = \sum_{i=0}^n J_i \mathbf{p}_i. \quad (3.2)$$

The relation $\mathbf{\Pi}_2 \mathbf{J} = \mathbf{L} \mathbf{\Pi}_2 \mathbf{E}$ implies

$$J_i = z_i E_1. \quad (3.3)$$

Let us suppose that $E_0 = 1$. Then E_1 and E_2 are determined by the orientation of the one-dimensional subspace \mathcal{E} with respect to the subspaces $\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$. Also since \mathcal{J} has codimension 1, there exist constants W_0, W_1, \dots, W_n , determined by the orientation of the n -dimensional subspace \mathcal{J} with respect to the subspaces $\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ such that

$$\sum_{i=0}^n W_i J_i = 0. \quad (3.4)$$

Let us suppose that $W_0 = 1$. Then we have

$$J_0 = - \sum_{i=1}^n W_i J_i = - \sum_{i=1}^n W_i E_i z_i, \quad (3.5)$$

which since $E_0 = 1$ implies $J_0 = -Y E_0$, with

$$Y = \sum_{i=1}^n \alpha_i z_i, \quad \text{where } \alpha_i = W_i E_i. \quad (3.6)$$

As the E_i and W_i are arbitrary constants, we see that Y can be any linear combination of the z_i . In particular, with $W_1 E_1 = 1$ and $W_i E_i = 0$ when $i \neq 1$ we obtain

$$Y = z_1. \quad (3.7)$$

As a second example consider a $Y(1)$ subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1, \quad (3.8)$$

where all the spaces $\mathcal{E}, \mathcal{J}, \mathcal{V}$, and \mathcal{P}_1 are all two-dimensional. Choose as our basis for \mathcal{K} two vectors \mathbf{p}_1 and \mathbf{p}_2 in \mathcal{V} and two vectors \mathbf{p}_3 and \mathbf{p}_4 in \mathcal{P}_1 . Then since \mathcal{E} is two-dimensional, there generically exist constants e_{13}, e_{14}, e_{23} and e_{24} such that

$$\mathbf{p}_1 + e_{13} \mathbf{p}_3 + e_{14} \mathbf{p}_4 \in \mathcal{E}, \quad \mathbf{p}_2 + e_{23} \mathbf{p}_3 + e_{24} \mathbf{p}_4 \in \mathcal{E}. \quad (3.9)$$

Also since \mathcal{J} is two-dimensional, there generically exist constants j_{31}, j_{32}, j_{41} and j_{42} such that

$$\mathbf{p}_3 + j_{31} \mathbf{p}_1 + j_{32} \mathbf{p}_2 \in \mathcal{J}, \quad \mathbf{p}_4 + j_{41} \mathbf{p}_1 + j_{42} \mathbf{p}_2 \in \mathcal{J}. \quad (3.10)$$

So the Y -problem is solved with vectors

$$\begin{aligned}
\mathbf{E} &= \mathbf{p}_1 + e_{13}\mathbf{p}_3 + e_{14}\mathbf{p}_4, \\
\mathbf{E}_1 &= \mathbf{p}_1, \quad \mathbf{E}_2 = e_{13}\mathbf{p}_3 + e_{14}\mathbf{p}_4, \\
\mathbf{J}_2 &= z_1(e_{13}\mathbf{p}_3 + e_{14}\mathbf{p}_4), \\
\mathbf{J} &= z_1[e_{13}(\mathbf{p}_3 + j_{31}\mathbf{p}_1 + j_{32}\mathbf{p}_2) + e_{14}(\mathbf{p}_4 + j_{41}\mathbf{p}_1 + e_{42}\mathbf{p}_2)], \\
\mathbf{J}_1 &= z_1[(e_{13}j_{31} + e_{14}j_{41})\mathbf{p}_1 + (e_{13}j_{32} + e_{42}j_{42})\mathbf{p}_2],
\end{aligned} \tag{3.11}$$

and is also solved with vectors

$$\begin{aligned}
\mathbf{E} &= \mathbf{p}_2 + e_{23}\mathbf{p}_3 + e_{24}\mathbf{p}_4, \\
\mathbf{E}_1 &= \mathbf{p}_2, \quad \mathbf{E}_2 = e_{23}\mathbf{p}_3 + e_{24}\mathbf{p}_4, \\
\mathbf{J}_2 &= z_1(e_{23}\mathbf{p}_3 + e_{24}\mathbf{p}_4), \\
\mathbf{J} &= z_1[e_{23}(\mathbf{p}_3 + j_{31}\mathbf{p}_1 + j_{32}\mathbf{p}_2) + e_{24}(\mathbf{p}_4 + j_{41}\mathbf{p}_1 + e_{42}\mathbf{p}_2)], \\
\mathbf{J}_1 &= z_1[(e_{23}j_{31} + e_{24}j_{41})\mathbf{p}_1 + (e_{23}j_{32} + e_{24}j_{42})\mathbf{p}_2].
\end{aligned} \tag{3.12}$$

From these equations it follows that $\mathbf{Y}(z_1)$ in this basis is the 2 by 2 matrix

$$\mathbf{Y}(z_1) = z_1 \mathbf{A}, \quad \text{with } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned}
a_{11} &= e_{13}j_{31} + e_{14}j_{41}, & a_{12} &= e_{13}j_{32} + e_{42}j_{42}, \\
a_{21} &= e_{23}j_{31} + e_{24}j_{41}, & a_{22} &= e_{23}j_{32} + e_{24}j_{42}.
\end{aligned} \tag{3.14}$$

As the coefficients $e_{13}, e_{14}, e_{23}, e_{24}, j_{31}, j_{32}, j_{41}$ and j_{42} can be any complex numbers we desire it follows that we can realize any desired complex matrix \mathbf{A} . By taking \mathcal{V}^I to be the one-dimensional space spanned by \mathbf{p}_1 and taking \mathcal{V}^O to be the one-dimensional space spanned by \mathbf{p}_2 we obtain a superfunction Y^S where the associated function takes the form

$$\mathbf{F}(z_1) = \begin{pmatrix} b_{11} & b_{12}/z_1 \\ b_{21}z_1 & b_{22} \end{pmatrix}, \tag{3.15}$$

in which the parameters b_{11}, b_{12}, b_{21} and b_{22} can be any complex numbers we choose.

As a third example consider a $Z(2)$ subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2, \tag{3.16}$$

where the subspaces $\mathcal{U}, \mathcal{E}, \mathcal{J}$ and \mathcal{P}_2 are all one-dimensional, while \mathcal{P}_1 is two-dimensional. Choose, as our basis for \mathcal{H} , 3 vectors $\mathbf{U}_0 \in \mathcal{U}$, $\mathbf{E}_0 \in \mathcal{E}$ and $\mathbf{J}_0 \in \mathcal{J}$, and take a vector \mathbf{P} as a basis for \mathcal{P}_2 . The coefficients P_U, P_E and P_J in the expansion

$$\mathbf{P} = P_U \mathbf{U}_0 + P_E \mathbf{E}_0 + P_J \mathbf{J}_0 \tag{3.17}$$

determine the orientation of \mathcal{P}_2 with respect to the subspaces \mathcal{U}, \mathcal{E} and \mathcal{J} . In the basis $\mathbf{U}_0, \mathbf{E}_0$, and \mathbf{J}_0 the equations

$$\mathbf{e} + \mathbf{E} = \mathbf{Q} + \alpha \mathbf{P}, \quad \mathbf{j} + \mathbf{J} = z_1 \mathbf{Q} + z_2 \alpha \mathbf{P}, \tag{3.18}$$

with

$$\mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{Q} \in \mathcal{P}_1, \quad (3.19)$$

take the form

$$\begin{aligned} \begin{pmatrix} e \\ E \\ 0 \end{pmatrix} &= \begin{pmatrix} Q_U \\ Q_E \\ Q_J \end{pmatrix} + \alpha \begin{pmatrix} P_U \\ P_E \\ P_J \end{pmatrix}, \\ \begin{pmatrix} j \\ 0 \\ J \end{pmatrix} &= z_1 \begin{pmatrix} Q_U \\ Q_E \\ Q_J \end{pmatrix} + z_2 \alpha \begin{pmatrix} P_U \\ P_E \\ P_J \end{pmatrix}, \end{aligned} \quad (3.20)$$

and since $\mathbf{Q} \in \mathcal{P}_1$ there exist constants W_U , W_E and W_J , which determine the orientation of \mathcal{P}_1 with respect to \mathcal{U} , \mathcal{E} and \mathcal{J} , such that

$$W_U Q_U + W_E Q_E + W_J Q_J = 0. \quad (3.21)$$

Hence we obtain the equations

$$\begin{aligned} W_U e + W_E E &= \alpha(W_U P_U + W_E P_E + W_J P_J) \equiv \alpha \mathbf{W} \cdot \mathbf{P}, \\ 0 &= z_1(E - \alpha P_E) + z_2 \alpha P_E, \\ j &= z_1(e - \alpha P_U) + z_2 \alpha P_U. \end{aligned} \quad (3.22)$$

Eliminating E and α from these equations gives $j = Ze$, with

$$Z = z_1 + \frac{(z_2 - z_1)W_U P_U}{\mathbf{W} \cdot \mathbf{P} + W_E P_E(z_2 - z_1)/z_1}. \quad (3.23)$$

In particular if the subspaces are oriented so that

$$\mathbf{W} \cdot \mathbf{P} = W_E P_E = -W_U P_U, \quad (3.24)$$

then (3.23) gives

$$Z = z_1^2/z_2, \quad (3.25)$$

which with $z_2 = 1$ produces the function z_1^2 and with $z_1 = 1$ produces the function $1/z_2$. Also, with $W_E P_E = 0$ we obtain

$$Z = z_1 + \frac{(z_2 - z_1)W_U P_U}{\mathbf{W} \cdot \mathbf{P}}, \quad (3.26)$$

which is a “weighted average” of z_1 and z_2 , $Z = w_1 z_1 + w_2 z_2$ with “weights” w_1 and w_2 that sum to 1 but which are not necessarily positive, nor even real.

4 Formulas for the associated functions

Following Section 12.8 of Milton (2002) a formula for the effective tensor \mathbf{Z} results by applying the operator $\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2$ (which projects on the space $\mathcal{U} \oplus \mathcal{J}$) to both sides of the constitutive law $\mathbf{e} + \mathbf{E} = \mathbf{L}^{-1}(\mathbf{j} + \mathbf{J})$. Solving the resulting equation,

$$\mathbf{e} = (\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{L}^{-1}(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)(\mathbf{j} + \mathbf{J}), \quad (4.1)$$

for $\mathbf{j} + \mathbf{J}$ gives

$$\mathbf{j} + \mathbf{J} = [(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{L}^{-1}(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)]^{-1}\mathbf{e}, \quad (4.2)$$

where the last inverse is to be taken on the subspace $\mathcal{U} \oplus \mathcal{J}$. By applying $\mathbf{\Gamma}_0$ to both sides of this equation we see that

$$\mathbf{Z} = \mathbf{\Gamma}_0[(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{L}^{-1}(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)]^{-1}\mathbf{\Gamma}_0, \quad (4.3)$$

which is the result given in (12.59) of Milton (2002).

Another formula for \mathbf{Z} follows from noting that for any arbitrary constant $z_0 \neq 0$,

$$[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})](\mathbf{e} + \mathbf{E}) = z_0\mathbf{e} + z_0\mathbf{E} - \mathbf{\Gamma}_1\mathbf{J} - z_0\mathbf{\Gamma}_1\mathbf{E} = z_0\mathbf{e}. \quad (4.4)$$

Solving this for $\mathbf{e} + \mathbf{E}$ gives

$$\mathbf{e} + \mathbf{E} = z_0[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})]^{-1}\mathbf{e}, \quad (4.5)$$

and applying $\mathbf{\Gamma}_0\mathbf{L}$ to both sides yields

$$\mathbf{j} = z_0\mathbf{\Gamma}_0\mathbf{L}[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})]^{-1}\mathbf{e}. \quad (4.6)$$

Thus we have a formula for the \mathbf{Z} operator,

$$\mathbf{Z} = z_0\mathbf{\Gamma}_0\mathbf{L}[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})]^{-1}\mathbf{\Gamma}_0 = z_0\mathbf{\Gamma}_0 + z_0\mathbf{\Gamma}_0(\mathbf{L} - z_0\mathbf{I})[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})]^{-1}\mathbf{\Gamma}_0, \quad (4.7)$$

where we have used the identity

$$\mathbf{\Gamma}_0 = z_0\mathbf{\Gamma}_0[z_0\mathbf{I} - \mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})]^{-1}\mathbf{\Gamma}_0, \quad (4.8)$$

obtained by applying $\mathbf{\Gamma}_0$ to both sides of (4.5). This formula (4.7) is a special case of the formula (12.60) given in Milton (2002), and is well known in different contexts (Kr ner 1977).

To obtain a formula for \mathbf{Y} notice that (2.6) and (2.8) imply that

$$0 = \mathbf{\Gamma}_2\mathbf{E}' = \mathbf{\Gamma}_2\mathbf{E}_1 + \mathbf{\Gamma}_2\mathbf{E}_2 = \mathbf{\Gamma}_2\mathbf{E}_1 + \mathbf{\Gamma}_2\mathbf{L}^{-1}\mathbf{\Pi}_2\mathbf{\Gamma}_2\mathbf{J}', \quad (4.9)$$

where the inverse of \mathbf{L} is to be taken on the subspace \mathcal{H} . Solving for \mathbf{J}' gives

$$\mathbf{J}' = -(\mathbf{\Gamma}_2\mathbf{L}^{-1}\mathbf{\Pi}_2\mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\mathbf{E}_1, \quad (4.10)$$

where the inverse is to be taken on the subspace \mathcal{J} . Then by applying $\mathbf{\Pi}_1$ to both sides of this equation and equating $\mathbf{\Pi}_1\mathbf{J}' = \mathbf{J}_1$ with $-\mathbf{Y}\mathbf{E}_1$ we obtain the desired formula

$$\mathbf{Y} = \mathbf{\Pi}_1\mathbf{\Gamma}_2(\mathbf{\Gamma}_2\mathbf{L}^{-1}\mathbf{\Pi}_2\mathbf{\Gamma}_2)^{-1}\mathbf{\Gamma}_2\mathbf{\Pi}_1, \quad (4.11)$$

for \mathbf{Y} , as given in formula (19.29) of Milton (2002).

Another formula for \mathbf{Y} is obtained by taking an arbitrary constant $z_0 \neq 0$, and defining

$$\mathbf{P}' = \mathbf{J}' - z_0\mathbf{E}'. \quad (4.12)$$

Applying $\mathbf{\Gamma}_1$ to both sides of (4.12) gives

$$\mathbf{\Gamma}_1\mathbf{P}' = -z_0\mathbf{E}' = -z_0(\mathbf{E}_1 + \mathbf{E}_2), \quad (4.13)$$

and applying Π_2 to both sides of (4.13) gives

$$\Pi_2 \mathbf{P}' = \mathbf{J}_2 - z_0 \mathbf{E}_2 = (\mathbf{L} - z_0 \mathbf{I}) \mathbf{E}_2. \quad (4.14)$$

Combining these results we see that \mathbf{P}' satisfies

$$[\Gamma_1 + z_0(\mathbf{L} - z_0 \mathbf{I})^{-1} \Pi_1] \mathbf{P}' = -z_0 \mathbf{E}_1. \quad (4.15)$$

Assuming that the operator $[\Gamma_1 + z_0(\mathbf{L} - z_0 \mathbf{I})^{-1} \Pi_1]$ is nonsingular this gives

$$\mathbf{P}' = -z_0 [\Gamma_1 + z_0(\mathbf{L} - z_0 \mathbf{I})^{-1} \Pi_1]^{-1} \mathbf{E}_1. \quad (4.16)$$

Applying $\Pi_1 = \mathbf{I} - \Pi_2$ to both sides yields

$$\mathbf{J}_1 - z_0 \mathbf{E}_1 = -(\mathbf{Y} + z_0 \mathbf{I}) \mathbf{E}_1 = -z_0 \Gamma_1 [\Gamma_1 + z_0(\mathbf{L} - z_0 \mathbf{I})^{-1} \Pi_1]^{-1} \mathbf{E}_1 \quad (4.17)$$

As this holds for all $\mathbf{E}_1 \in \mathcal{V}$ we obtain the formula

$$\mathbf{Y} = -z_0 \Pi_1 + z_0 \Gamma_1 [\Gamma_1 + z_0(\mathbf{L} - z_0 \mathbf{I})^{-1} \Pi_1]^{-1} \Pi_1 \quad (4.18)$$

which is a special case of the formula (19.37) obtained in Section 19.5 of Milton (2002).

5 Multiplying superfunctions

Multiplying superfunctions is similar the way electrical circuits, each with $2m$ terminal can be combined. An example is shown in Figure 5.

Suppose we have two superfunctions, $(F^s)'$ and $(F^s)''$:

$$\begin{aligned} \mathcal{K}' &= \mathcal{E}' \oplus \mathcal{J}' = (\mathcal{V}^I)' \oplus (\mathcal{V}^O)' \oplus \mathcal{H}' \quad \text{with } \mathcal{H}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_j, \\ \mathcal{K}'' &= \mathcal{E}'' \oplus \mathcal{J}'' = (\mathcal{V}^I)'' \oplus (\mathcal{V}^O)'' \oplus \mathcal{H}'' \quad \text{with } \mathcal{H}'' = \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_k, \end{aligned} \quad (5.1)$$

where the spaces $(\mathcal{V}^I)', (\mathcal{V}^O)', (\mathcal{V}^I)'', (\mathcal{V}^O)''$ all have the same dimension m . To take their product one needs to first find two nonsingular linear operators \mathbf{M}^E and \mathbf{M}^J which map $(\mathcal{V}^O)'$ to $(\mathcal{V}^I)''$. The resulting product superfunction

$$F^s = (F^s)' \times_{\mathbf{M}} (F^s)'', \quad (5.2)$$

is the subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = (\mathcal{V}^I)' \oplus (\mathcal{V}^O)'' \oplus \mathcal{H}, \quad (5.3)$$

where

$$\mathcal{H} = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_j \oplus \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_k, \quad (5.4)$$

and the operator \mathbf{L} acting on \mathcal{H} is

$$\mathbf{L} = \sum_{i=1}^j z'_i \Lambda'_i + \sum_{\ell=1}^k z''_{\ell} \Lambda''_{\ell}, \quad (5.5)$$

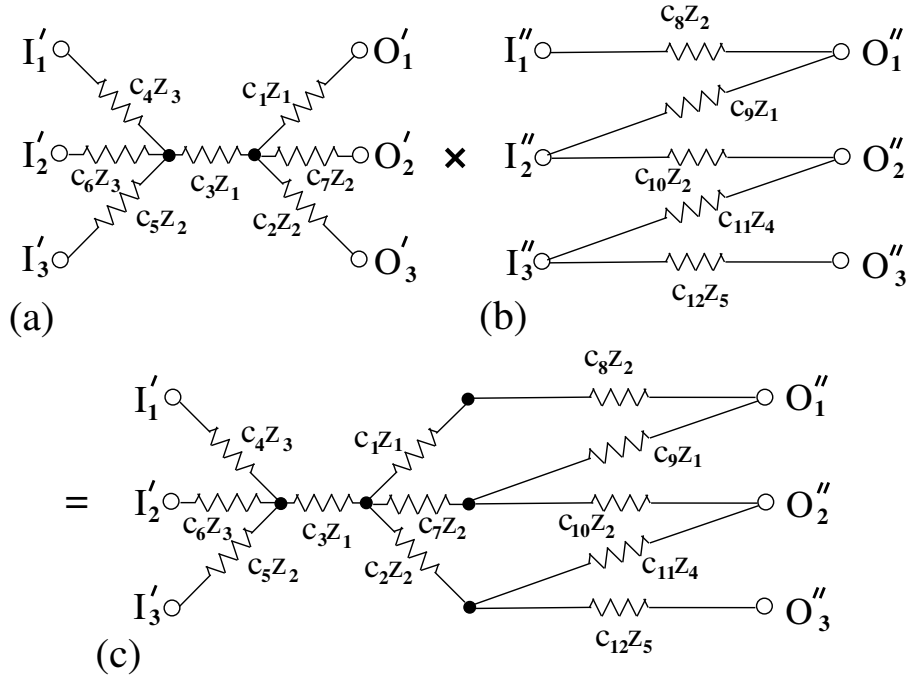


Figure 5: Multiplying superfunctions is like hooking networks, with an equal number of input and output terminals, together in series. Shown in (a) and (b) are 6 terminal electrical networks, each (along with their respective tree-like battery configurations on the opposite side of the circuit board that are not shown here) represent a superfunction as the terminals have been divided into input terminals (I'_1, I'_2 , and I'_3 for the circuit (a), and I''_1, I''_2 , and I''_3 for the circuit (b)) and output terminals (O'_1, O'_2 , and O'_3 for the circuit (a), and O''_1, O''_2 , and O''_3 for the circuit (b)). The product superfunction is the 6 terminal electrical network (along with its tree-like battery configurations on the opposite side of the circuit board) shown in (c). Note there is some flexibility in how one multiplies superfunctions: instead of connecting the terminals O'_i with I''_i for $i = 1, 2, 3$, one could for example, connect O'_1, O'_2 , and O'_3 with any permutation of I''_1, I''_2 and I''_3 . This is why, when taking a product, one needs to specify the maps (\mathbf{M}^E and \mathbf{M}^J) one is using between the output space of one superfunction, and the input space of the second superfunction by which one is multiplying it.

in which Λ'_i and Λ''_ℓ are the projections onto \mathcal{P}'_i and \mathcal{P}''_ℓ . A vector \mathbf{E} is in \mathcal{E} if and only if we can find vectors

$$\begin{aligned}\mathbf{E}' &= (\mathbf{E}^I)' + (\mathbf{E}^O)' + \mathbf{E}'_2 \in \mathcal{E}', \\ \mathbf{E}'' &= (\mathbf{E}^I)'' + (\mathbf{E}^O)'' + \mathbf{E}''_2 \in \mathcal{E}'',\end{aligned}\tag{5.6}$$

such that

$$(\mathbf{E}^I)'' = \mathbf{M}^E(\mathbf{E}^O)', \quad \mathbf{E} = (\mathbf{E}^I)' + (\mathbf{E}^O)'' + \mathbf{E}'_2 + \mathbf{E}''_2,\tag{5.7}$$

with

$$(\mathbf{E}^I)' \in (\mathcal{V}^I)', \quad (\mathbf{E}^O)' \in (\mathcal{V}^O)', \quad \mathbf{E}'_2 \in \mathcal{H}', \quad (\mathbf{E}^I)'' \in (\mathcal{V}^I)'', \quad (\mathbf{E}^O)'' \in (\mathcal{V}^O)'', \quad \mathbf{E}''_2 \in \mathcal{H}''.\tag{5.8}$$

A vector \mathbf{J} is in \mathcal{J} if and only if we can find vectors

$$\begin{aligned}\mathbf{J}' &= (\mathbf{J}^I)' + (\mathbf{J}^O)' + \mathbf{J}'_2 \in \mathcal{J}', \\ \mathbf{J}'' &= (\mathbf{J}^I)'' + (\mathbf{J}^O)'' + \mathbf{J}''_2 \in \mathcal{J}'',\end{aligned}\tag{5.9}$$

such that

$$(\mathbf{J}^I)'' = \mathbf{M}^J(\mathbf{J}^O)', \quad \mathbf{J} = (\mathbf{J}^I)' + (\mathbf{J}^O)'' + \mathbf{J}'_2 + \mathbf{J}''_2,\tag{5.10}$$

with

$$(\mathbf{J}^I)' \in (\mathcal{V}^I)', \quad (\mathbf{J}^O)' \in (\mathcal{V}^O)', \quad \mathbf{J}'_2 \in \mathcal{H}', \quad (\mathbf{J}^I)'' \in (\mathcal{V}^I)'', \quad (\mathbf{J}^O)'' \in (\mathcal{V}^O)'', \quad \mathbf{J}''_2 \in \mathcal{H}''.\tag{5.11}$$

To ensure that the two spaces \mathcal{E} and \mathcal{J} are independent we need to make the technical assumption that \mathbf{M}^E and \mathbf{M}^J are chosen so that the operator \mathbf{A} mapping $(\mathcal{V}^O)'$ to $(\mathcal{V}^I)''$, defined by

$$\mathbf{A} = \mathbf{M}^E(\Pi^O)'\Gamma'_1 - (\Pi^I)''\Gamma''_1[\mathbf{M}^E(\Pi^O)'\Gamma'_1 + \mathbf{M}^J(\Pi^O)'\Gamma'_2],\tag{5.12}$$

is nonsingular (i.e. the null-space of the operator contains only the zero vector). Our aim is to show that if \mathbf{A} is nonsingular and

$$\mathbf{E} = (\mathbf{E}^I)' + (\mathbf{E}^O)'' + \mathbf{E}'_2 + \mathbf{E}''_2 = \mathbf{J} = (\mathbf{J}^I)' + (\mathbf{J}^O)'' + \mathbf{J}'_2 + \mathbf{J}''_2, \quad \text{with } \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}\tag{5.13}$$

then $\mathbf{E} = \mathbf{J} = 0$. First note that by resolving (5.13) into components in the spaces $(\mathcal{V}^I)'$, $(\mathcal{V}^I)''$, \mathcal{H}' , and \mathcal{H}'' we obtain

$$(\mathbf{E}^I)' = (\mathbf{J}^I)', \quad (\mathbf{E}^O)'' = (\mathbf{J}^O)'', \quad \mathbf{E}'_2 = \mathbf{J}'_2, \quad \mathbf{E}''_2 = \mathbf{J}''_2.\tag{5.14}$$

Also since $\mathbf{E} \in \mathcal{E}$ and $\mathbf{J} \in \mathcal{J}$ there exist vectors $(\mathbf{E}^O)', (\mathbf{J}^O)' \in (\mathcal{V}^O)'$ and $(\mathbf{E}^I)'', (\mathbf{J}^I)'' \in (\mathcal{V}^I)''$ such that (5.6) and (5.9) hold. Since $\mathcal{E}' \cap \mathcal{J}' = \{0\}$ and $\mathcal{E}'' \cap \mathcal{J}'' = \{0\}$ it follows that

$$\mathbf{P} \equiv (\mathbf{E}^O)' - (\mathbf{J}^O)' = \mathbf{E}' - \mathbf{J}' \neq 0 \quad \text{or} \quad \mathbf{E}' = \mathbf{J}' = 0,\tag{5.15}$$

and

$$\mathbf{Q} \equiv (\mathbf{E}^I)'' - (\mathbf{J}^I)'' = \mathbf{E}'' - \mathbf{J}'' \neq 0 \quad \text{or} \quad \mathbf{E}'' = \mathbf{J}'' = 0.\tag{5.16}$$

Now we have

$$\begin{aligned}(\Pi^O)'\Gamma'_1\mathbf{P} &= (\Pi^O)'\mathbf{E}' = (\mathbf{E}^O)', & (\Pi^O)'\Gamma'_2\mathbf{P} &= -(\Pi^O)'\mathbf{J}' = -(\mathbf{J}^O)' \\ (\Pi^I)''\Gamma''_1\mathbf{Q} &= (\Pi^I)''\mathbf{E}'' = (\mathbf{E}^I)'', & (\Pi^I)''\Gamma''_2\mathbf{Q} &= -(\Pi^I)''\mathbf{J}'' = -(\mathbf{J}^I)''.\end{aligned}\tag{5.17}$$

Since $(\mathbf{E}^I)'' = \mathbf{M}^E(\mathbf{E}^O)'$ and $(\mathbf{J}^I)'' = \mathbf{M}^J(\mathbf{J}^O)'$ we get from the first pair of equations in (5.17) the result that

$$(\mathbf{E}^I)'' = \mathbf{M}^E(\Pi^O)'\Gamma'_1\mathbf{P}, \quad (\mathbf{J}^I)'' = -\mathbf{M}^J(\Pi^O)'\Gamma'_2\mathbf{P},\tag{5.18}$$

which implies

$$\mathbf{Q} = [\mathbf{M}^E(\Pi^O)'\Gamma'_1 + \mathbf{M}^J(\Pi^O)'\Gamma'_2]\mathbf{P}.\tag{5.19}$$

Substituting this back in the second pair of equations in (5.17), and using (5.18), gives

$$\begin{aligned}(\Pi^I)''\Gamma''_1[\mathbf{M}^E(\Pi^O)'\Gamma'_1 + \mathbf{M}^J(\Pi^O)'\Gamma'_2]\mathbf{P} &= \mathbf{M}^E(\Pi^O)'\Gamma'_1\mathbf{P} \\ (\Pi^I)''\Gamma''_2[\mathbf{M}^E(\Pi^O)'\Gamma'_1 + \mathbf{M}^J(\Pi^O)'\Gamma'_2]\mathbf{P} &= \mathbf{M}^J(\Pi^O)'\Gamma'_2\mathbf{P}.\end{aligned}\tag{5.20}$$

These two equations are not independent since by adding them we obtain

$$[\mathbf{M}^E(\boldsymbol{\Pi}^O)' \boldsymbol{\Gamma}'_1 + \mathbf{M}^J(\boldsymbol{\Pi}^O)' \boldsymbol{\Gamma}'_2] \mathbf{P} = \mathbf{M}^E(\boldsymbol{\Pi}^O)' \boldsymbol{\Gamma}'_1 \mathbf{P} + \mathbf{M}^J(\boldsymbol{\Pi}^O)' \boldsymbol{\Gamma}'_2 \mathbf{P} \quad (5.21)$$

which is obviously satisfied. Also the first equation in (5.20) says \mathbf{P} is in the null space of \mathbf{A} , which by our assumption implies $\mathbf{P} = 0$. Then (5.19) implies $\mathbf{Q} = 0$ and this rules out the first possibilities in (5.15) and (5.16), implying $\mathbf{E}' = \mathbf{J}' = 0$ and $\mathbf{E}'' = \mathbf{J}'' = 0$. We conclude that $\mathbf{E} = \mathbf{J} = 0$.

To check that the space $\mathcal{E} \oplus \mathcal{J}$ spans $(\mathcal{V}^I)' \oplus (\mathcal{V}^O)'' \oplus \mathcal{H}$, we just need to count dimensions. The dimension of the space on the right is $2m + \dim(\mathcal{H})$. The dimension of \mathcal{E} according to (5.6) is $\dim(\mathcal{E}') + \dim(\mathcal{E}'') - m$ because of the m constraints $(\mathbf{E}^I)'' = \mathbf{M}^E(\mathbf{E}^O)'$. Similarly the dimension of \mathcal{J} is $\dim(\mathcal{J}') + \dim(\mathcal{J}'') - m$. Adding these up, we get the dimension of $\mathcal{E} \oplus \mathcal{J}$ is $\dim \mathcal{K}' + \dim \mathcal{K}'' - 2m = 2m + \dim(\mathcal{H}') + \dim(\mathcal{H}'') = 2m + \dim(\mathcal{H})$.

Let \mathbf{F}' and \mathbf{F}'' be the functions associated with the superfunctions $(F^s)'$ and $(F^s)''$. Given operators

$$\mathbf{L}' = \sum_{i=1}^j z'_i \boldsymbol{\Lambda}'_i, \quad \mathbf{L}'' = \sum_{i=1}^k z''_i \boldsymbol{\Lambda}''_i, \quad (5.22)$$

where $\boldsymbol{\Lambda}'_i$ projects onto \mathcal{P}'_i and $\boldsymbol{\Lambda}''_i$ projects onto \mathcal{P}''_i , and given input fields $(\mathbf{E}^I)'$ and $(\mathbf{J}^I)'$ we can calculate

$$\begin{aligned} \begin{pmatrix} (\mathbf{E}^O)' \\ (\mathbf{J}^O)' \end{pmatrix} &= \mathbf{F}' \begin{pmatrix} (\mathbf{E}^I)' \\ (\mathbf{J}^I)' \end{pmatrix}, \\ (\mathbf{E}^I)' &= \mathbf{M}^E(\mathbf{E}^O)'', \quad (\mathbf{J}^I)' = \mathbf{M}^J(\mathbf{J}^O)'', \\ \begin{pmatrix} (\mathbf{E}^O)'' \\ (\mathbf{J}^O)'' \end{pmatrix} &= \mathbf{F}'' \begin{pmatrix} (\mathbf{E}^I)'' \\ (\mathbf{J}^I)'' \end{pmatrix}. \end{aligned} \quad (5.23)$$

From the knowledge of $(\mathbf{E}^O)'$ and $(\mathbf{E}^I)'$, and of $(\mathbf{E}^O)''$ and $(\mathbf{E}^I)''$, we can calculate the fields \mathbf{E}' , \mathbf{E}'' , \mathbf{J}' , and \mathbf{J}'' of the form (5.6) and (5.9) solving the Y' problem and the Y'' problem:

$$\begin{aligned} \mathbf{E}' &\in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}', \quad \mathbf{J}'_1 = \mathbf{L}' \mathbf{E}'_1, \\ \mathbf{E}'' &\in \mathcal{E}'', \quad \mathbf{J}'' \in \mathcal{J}'', \quad \mathbf{J}''_1 = \mathbf{L}'' \mathbf{E}''_1. \end{aligned} \quad (5.24)$$

Then the fields \mathbf{E} and \mathbf{J} given by (5.7) and (5.10) solve the Y problem in the space \mathcal{K} , and the function associated to the superfunction F^s is given by the product rule

$$\mathbf{F} = \mathbf{F}' \begin{pmatrix} \mathbf{M}^E & 0 \\ 0 & \mathbf{M}^J \end{pmatrix} \mathbf{F}'' \quad (5.25)$$

Let us choose a basis $(\mathbf{v}_1^I)'', (\mathbf{v}_2^I)'', \dots, (\mathbf{v}_m^I)''$ for $(\mathcal{V}^I)''$, choose a basis $(\mathbf{v}_1^O)'', (\mathbf{v}_2^O)'', \dots, (\mathbf{v}_m^O)''$ for $(\mathcal{V}^O)''$, take $\mathbf{M}^E(\mathbf{v}_1^O)'', \mathbf{M}^E(\mathbf{v}_2^O)'', \dots, \mathbf{M}^E(\mathbf{v}_m^O)''$ as our basis for $(\mathcal{V}^I)'$, and choose a basis $(\mathbf{v}_1^O)', (\mathbf{v}_2^O)', \dots, (\mathbf{v}_m^O)'$ for $(\mathcal{V}^O)'$. Then the operator \mathbf{M}^E is represented as the identity matrix in the basis. Let us also choose the operator \mathbf{M}^J so it is represented by *minus* the identity matrix in this basis. Then in this basis the relation (5.25) takes the form

$$\mathbf{F} = \mathbf{F}' \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \mathbf{F}'' \quad (5.26)$$

Note that we could have avoided this slightly awkward multiplication rule if we had replaced the definition (2.17) of the associated function by

$$\begin{pmatrix} \mathbf{E}^O \\ -\mathbf{J}^O \end{pmatrix} = \mathbf{F} \begin{pmatrix} \mathbf{E}^I \\ \mathbf{J}^I \end{pmatrix}. \quad (5.27)$$

Then the multiplication rule (with this choice of \mathbf{M}^E and \mathbf{M}^J) would have become simply $\mathbf{F} = \mathbf{F}'\mathbf{F}''$. We chose not to do this in the interest of preserving the symmetric roles of the spaces \mathcal{E} and \mathcal{J} in the definition of the function associated with the superfunction.

In passing, let us suppose there is an inner product on the vector spaces \mathcal{K}' and \mathcal{K}'' , and that the sets of spaces $\{\mathcal{E}', \mathcal{J}'\}$, $\{(\mathcal{V}^I)', (\mathcal{V}^O)', \mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_j\}$, $\{\mathcal{E}'', \mathcal{J}''\}$, $\{(\mathcal{V}^I)'', (\mathcal{V}^O)'', \mathcal{P}''_1, \mathcal{P}''_2, \dots, \mathcal{P}''_k\}$ all contain mutually orthogonal spaces. For any two fields

$$\mathbf{P} = \mathbf{P}^I + \mathbf{P}^O + \mathbf{P}' + \mathbf{P}'', \quad \mathbf{Q} = \mathbf{Q}^I + \mathbf{Q}^O + \mathbf{Q}' + \mathbf{Q}'', \quad (5.28)$$

in the vector space \mathcal{K} , with

$$\mathbf{P}^I, \mathbf{Q}^I \in (\mathcal{V}^I)', \quad \mathbf{P}^O, \mathbf{Q}^O \in (\mathcal{V}^O)'', \quad \mathbf{P}', \mathbf{Q}' \in \mathcal{H}', \quad \mathbf{P}'', \mathbf{Q}'' \in \mathcal{H}'', \quad (5.29)$$

let us define the inner product of them to be

$$(\mathbf{P}, \mathbf{Q}) = (\mathbf{P}^I, \mathbf{Q}^I)' + (\mathbf{P}^O, \mathbf{Q}^O)'' + (\mathbf{P}', \mathbf{Q}')' + (\mathbf{P}'', \mathbf{Q}'')'', \quad (5.30)$$

in which $(\ , \)'$ and $(\ , \)''$ denote the inner product on the spaces \mathcal{K}' and \mathcal{K}'' respectively. It is immediately clear from this definition that the subspaces $(\mathcal{V}^I)', (\mathcal{V}^O)'', \mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_j, \mathcal{P}''_1, \mathcal{P}''_2, \dots, \mathcal{P}''_k$ are mutually orthogonal in the new superfunction. Now take a field $\mathbf{E} \in \mathcal{E}$ and $\mathbf{J} \in \mathcal{J}$. By the definition of these subspaces there must exist fields $\mathbf{E}' \in \mathcal{E}'$ and $\mathbf{E}'' \in \mathcal{E}''$ such that (5.6) to (5.8) hold, and fields $\mathbf{J}' \in \mathcal{J}'$, $\mathbf{J}'' \in \mathcal{J}''$ such that (5.9) to (5.11) hold. Consequently we have

$$\begin{aligned} (\mathbf{J}, \mathbf{E}) &= (\mathbf{J}' + \mathbf{J}'' - (\mathbf{J}^O)' - (\mathbf{J}^I)'', \mathbf{E}' + \mathbf{E}'' - (\mathbf{E}^O)' - (\mathbf{E}^I)'') \\ &= ((\mathbf{J}^O)', (\mathbf{E}^O)')' + ((\mathbf{J}^I)'', (\mathbf{E}^I)'')'' - (\mathbf{J}', (\mathbf{E}^O)')' - (\mathbf{J}'', (\mathbf{E}^I)'')'' - ((\mathbf{J}^O)', \mathbf{E}')' - ((\mathbf{J}^I)'', \mathbf{E}'')'' \\ &= -((\mathbf{J}^O)', (\mathbf{E}^O)')' - ((\mathbf{J}^I)'', (\mathbf{E}^I)'')'' \\ &= -((\mathbf{J}^O)', (\mathbf{E}^O)')' - (\mathbf{M}^J (\mathbf{J}^O)', \mathbf{M}^E (\mathbf{E}^O)')'' \\ &= -((\mathbf{J}^O)', (\mathbf{E}^O)')' - ((\mathbf{M}^E)^\dagger \mathbf{M}^J (\mathbf{J}^O)', (\mathbf{E}^O)')', \end{aligned} \quad (5.31)$$

in which $(\mathbf{M}^E)^\dagger$ is the adjoint of \mathbf{M}^E . So we see that the spaces \mathcal{J} and \mathcal{E} will be orthogonal if we choose

$$(\mathbf{M}^E)^\dagger \mathbf{M}^J = -\mathbf{I}. \quad (5.32)$$

Note that the orthogonality of the spaces \mathcal{J} and \mathcal{E} immediately implies that they have no nonzero vector in their intersection.

In the case of nonorthogonal subspace collections, we are free to choose the maps \mathbf{M}^E and \mathbf{M}^J that map $(\mathcal{V}^O)'$ to $(\mathcal{V}^I)''$, so long as they and the map \mathbf{A} are nonsingular. However, in view of (5.32), it would be quite natural to restrict our definition of multiplication by requiring that $\mathbf{M}^J = -\mathbf{M}^E$, i.e. one can pick a nonsingular map \mathbf{M} mapping $(\mathcal{V}^O)'$ to $(\mathcal{V}^I)''$ and set

$$\mathbf{M}^E = \mathbf{M}, \quad \mathbf{M}^J = -\mathbf{M}. \quad (5.33)$$

With this choice, subtracting the equations in (5.20) gives

$$(\mathbf{\Pi}^I)''(\mathbf{\Gamma}_1'' - \mathbf{\Gamma}_2'')\mathbf{M}(\mathbf{\Pi}^O)'(\mathbf{\Gamma}_1' - \mathbf{\Gamma}_2')\mathbf{P} = \mathbf{M}\mathbf{P} \quad (5.34)$$

Returning to the case where the subspaces are orthogonal, (5.32) is satisfied if $\mathbf{M}\mathbf{M}^\dagger = -\mathbf{I}$. An alternative way to see that \mathcal{J} and \mathcal{E} have no nonzero vector in their intersection is as follows. Choose an orthonormal basis $(\mathbf{v}_1^O)', (\mathbf{v}_2^O)', \dots, (\mathbf{v}_m^O)'$ for $(\mathcal{V}^O)'$ and take $\mathbf{M}^E = -\mathbf{M}^J$ as a map such that $\mathbf{M}^E(\mathbf{v}_1^O)', \mathbf{M}^E(\mathbf{v}_2^O)', \dots, \mathbf{M}^E(\mathbf{v}_m^O)'$ form an orthonormal basis for $(\mathcal{V}^I)''$. Then the operator \mathbf{M}^E is represented as the identity matrix in the basis, and \mathbf{M}^J is represented by $-\mathbf{I}$. Now, recalling the definition of the norm $|\mathbf{Q}| = (\mathbf{Q}, \mathbf{Q})^{1/2}$ of a vector \mathbf{Q} recall that the action of the operators $(\mathbf{\Pi}^O)'$, $(\mathbf{\Pi}^I)''$ cannot increase the norm of a vector, while $\mathbf{\Gamma}_1' - \mathbf{\Gamma}_2'$ and $\mathbf{\Gamma}_1'' - \mathbf{\Gamma}_2''$ preserve the norm (as can be seen if we take a basis where these are diagonal). Hence (5.34) can be satisfied only when there is a $\mathbf{P} \in (\mathcal{V}^O)'$ such that

$$(\mathbf{\Gamma}_1' - \mathbf{\Gamma}_2')\mathbf{P} \in (\mathcal{V}^O)' \quad (\mathbf{\Gamma}_1'' - \mathbf{\Gamma}_2'')\mathbf{M}\mathbf{P} \in (\mathcal{V}^I)''. \quad (5.35)$$

Then as $\mathbf{\Gamma}_1' + \mathbf{\Gamma}_2' = \mathbf{I}$ and $\mathbf{\Gamma}_1'' + \mathbf{\Gamma}_2'' = \mathbf{I}$ we obtain

$$(\mathbf{\Gamma}_1' + \mathbf{\Gamma}_2')\mathbf{P} \in (\mathcal{V}^O)' \quad (\mathbf{\Gamma}_1'' + \mathbf{\Gamma}_2'')\mathbf{M}\mathbf{P} \in (\mathcal{V}^I)''. \quad (5.36)$$

Adding and subtracting (5.35) and (5.36) then implies

$$\mathbf{\Gamma}_1'\mathbf{P} \in (\mathcal{V}^O)', \quad \mathbf{\Gamma}_2'\mathbf{P} \in (\mathcal{V}^O)', \quad \mathbf{\Gamma}_1''\mathbf{P} \in (\mathcal{V}^I)'', \quad \mathbf{\Gamma}_2''\mathbf{P} \in (\mathcal{V}^I)'' \quad (5.37)$$

which is excluded by our assumption that \mathcal{V}' has no vector in common with \mathcal{E}' or \mathcal{J}' and that \mathcal{V}'' has no vector in common with \mathcal{E}'' or \mathcal{J}'' .

6 Multiplicative identity superfunctions

Suppose we are given nonsingular maps \mathbf{M}^E and \mathbf{M}^J which map the m -dimensional space $(\mathcal{V}^O)'$ to the m -dimensional space $(\mathcal{V}^I)''$. Let \mathcal{K}'' denote the $2m$ -dimensional space

$$\mathcal{K}'' = (\mathcal{V}^O)' \oplus (\mathcal{V}^I)''. \quad (6.1)$$

Within this space define \mathcal{E}'' as the subspace consisting of all vectors of the form $\mathbf{E} = \mathbf{v} + (\mathbf{M}^E)^{-1}\mathbf{v}$ with $\mathbf{v} \in (\mathcal{V}^I)''$ and define \mathcal{J}'' as the subspace consisting of all vectors of the form $\mathbf{J} = \mathbf{w} + (\mathbf{M}^J)^{-1}\mathbf{w}$ with $\mathbf{w} \in (\mathcal{V}^I)''$. If these subspaces have a vector in common then

$$\mathbf{v} + (\mathbf{M}^E)^{-1}\mathbf{v} = \mathbf{w} + (\mathbf{M}^J)^{-1}\mathbf{w}, \quad \text{i.e., } \mathbf{v} - \mathbf{w} = (\mathbf{M}^J)^{-1}\mathbf{w} - (\mathbf{M}^E)^{-1}\mathbf{v}. \quad (6.2)$$

In this last equation the fields on the left and on the right lie respectively in $(\mathcal{V}^I)''$ and $(\mathcal{V}^O)'$. As the intersection of these subspaces consists of only the zero vector, we conclude that both sides must be zero, i.e. $\mathbf{w} = \mathbf{v}$ and

$$\mathbf{u} \equiv (\mathbf{M}^E)^{-1}\mathbf{v} = (\mathbf{M}^J)^{-1}\mathbf{v} \quad (6.3)$$

Thus, $\mathbf{M}^E \mathbf{u} = \mathbf{v} = \mathbf{M}^J \mathbf{u}$ and if we assume that $\mathbf{M}^J - \mathbf{M}^E$ is nonsingular, then $0 = \mathbf{u} = \mathbf{v} = \mathbf{w}$. So under this assumption the subspaces have only the zero vector in their intersection. Then, since they each have dimension m we conclude that

$$\mathcal{K}'' = (\mathcal{V}^O)' \oplus (\mathcal{V}^I)'' = \mathcal{E}'' \oplus \mathcal{J}'', \quad (6.4)$$

which defines a superfunction $(F^s)''$ in which \mathcal{H} is empty.

We now look at the associated superfunction problem. As the space \mathcal{H} is empty, if we are given vectors \mathbf{E}^I and \mathbf{J}^I in the input space $(\mathcal{V}^I)''$ the superfunction problem then consists of finding vectors \mathbf{E}^O and \mathbf{J}^O in the output space $(\mathcal{V}^O)'$ such that

$$\mathbf{E}^I + \mathbf{E}^O \in \mathcal{E}'', \quad \mathbf{J}^I + \mathbf{J}^O \in \mathcal{J}''. \quad (6.5)$$

From our definition of the subspaces \mathcal{E}'' and \mathcal{J}'' we immediately see that the superfunction problem is solved with fields

$$\mathbf{E}^0 = (\mathbf{M}^E)^{-1} \mathbf{E}^I, \quad \mathbf{J}^0 = (\mathbf{M}^J)^{-1} \mathbf{J}^I, \quad (6.6)$$

implying, through (2.17), that the associated function is

$$\mathbf{F}'' = \begin{pmatrix} (\mathbf{M}^E)^{-1} & 0 \\ 0 & \mathbf{M}^{J^{-1}} \end{pmatrix}. \quad (6.7)$$

So if we take another superfunction $(F^s)'$ and multiply it by this superfunction $(F^s)''$, the product rule (5.25) implies that the resulting superfunction F^s has the associated function

$$\mathbf{F} = \mathbf{F}'. \quad (6.8)$$

We conclude that this superfunction $(F^s)''$ is the multiplicative identity, when multiplication is defined with the maps \mathbf{M}^E and \mathbf{M}^J .

7 Addition of Y -subspace collections and embeddings

Adding superfunctions is similar the way electrical circuits, each with n terminals can be combined. An example is shown in Figure 6.

Suppose we have $Y(j)$ and $Y(k)$ subspace collections:

$$\begin{aligned} \mathcal{K}' &= \mathcal{E}' \oplus \mathcal{J}' = \mathcal{V}' \oplus \mathcal{H}' \quad \text{with} \quad \mathcal{H}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_j, \\ \mathcal{K}'' &= \mathcal{E}'' \oplus \mathcal{J}'' = \mathcal{V}'' \oplus \mathcal{H}'' \quad \text{with} \quad \mathcal{H}'' = \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_k, \end{aligned} \quad (7.1)$$

where the spaces \mathcal{V}' and \mathcal{V}'' have the same dimension n . To define the sum of the subspace collections we need to introduce another n -dimensional space \mathcal{V} and nonsingular operators \mathbf{S}' and \mathbf{S}'' which respectively map \mathcal{V}' and \mathcal{V}'' to \mathcal{V} . Then the sum of the subspace collections

$$\mathcal{K} = \mathcal{K}' +_{\{\mathbf{S}', \mathbf{S}''\}} \mathcal{K}'' \quad (7.2)$$

is the subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{H}, \quad (7.3)$$

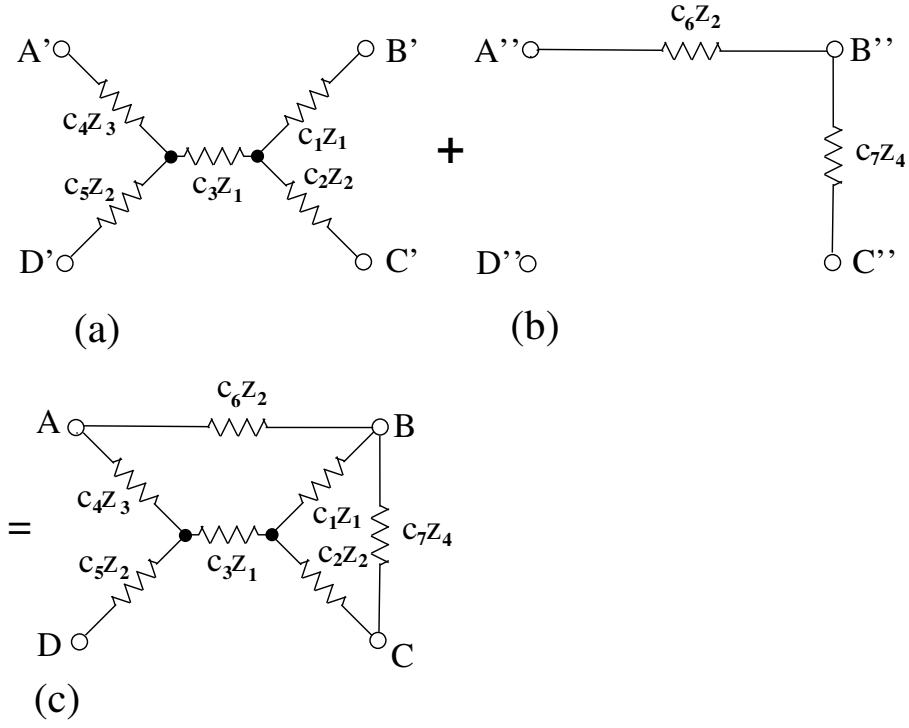


Figure 6: Adding Y -subspace collections is like hooking networks together in parallel. The 4 terminal networks in (a) and (b), each representing (along with their respective tree-like battery configurations on the opposite side of the circuit board that are not shown here) $Y(3)$ and $Y(2)$ subspace collections, are added together to form the 4 terminal network in (c) which is a $Y(4)$ subspace collection. Note that the circuit in (b) is really only a 3 terminal network, so it has been embedded in a 4 terminal network (with no electrical connections to the 4th terminal). Also note there is some flexibility in how one adds together these subspace collections: we connected the terminals A', B', C' , and D' , to respectively the terminals A'', B'', C'' , and D'' , but we could have connected them to any permutation of these terminals. This flexibility is reflected in the need to introduce nonsingular operators \mathbf{S}' and \mathbf{S}'' which respectively map \mathcal{V}' and \mathcal{V}'' to \mathcal{V} , before addition can be defined.

where

$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_j \oplus \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_k. \quad (7.4)$$

Here a field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, with $\mathbf{E}_1 \in \mathcal{V}$ and $\mathbf{E}_2 \in \mathcal{H}$, is in \mathcal{E} if and only if there exist fields

$$\mathbf{E}' = \mathbf{E}'_1 + \mathbf{E}'_2 \in \mathcal{E}', \quad \mathbf{E}'' = \mathbf{E}''_1 + \mathbf{E}''_2 \in \mathcal{E}'', \quad (7.5)$$

with

$$\mathbf{E}'_1 \in \mathcal{V}', \quad \mathbf{E}'_2 \in \mathcal{H}', \quad \mathbf{E}''_1 \in \mathcal{V}'', \quad \mathbf{E}''_2 \in \mathcal{H}'', \quad (7.6)$$

such that

$$\mathbf{S}'\mathbf{E}'_1 = \mathbf{S}''\mathbf{E}''_1 = \mathbf{E}_1. \quad (7.7)$$

Also a field $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, with $\mathbf{J}_1 \in \mathcal{V}$ and $\mathbf{J}_2 \in \mathcal{H}$, is in \mathcal{J} if and only if there exist fields

$$\mathbf{J}' = \mathbf{J}'_1 + \mathbf{J}'_2 \in \mathcal{E}', \quad \mathbf{J}'' = \mathbf{J}''_1 + \mathbf{J}''_2 \in \mathcal{E}'', \quad (7.8)$$

with

$$\mathbf{J}'_1 \in \mathcal{V}', \quad \mathbf{J}'_2 \in \mathcal{H}', \quad \mathbf{J}''_1 \in \mathcal{V}'', \quad \mathbf{J}''_2 \in \mathcal{H}'', \quad (7.9)$$

such that

$$\mathbf{S}'\mathbf{J}'_1 + \mathbf{S}''\mathbf{J}''_1 = \mathbf{J}_1. \quad (7.10)$$

So given $\mathbf{E}_1 \in \mathcal{V}$, we let $\mathbf{E}'_1 = (\mathbf{S}')^{-1}\mathbf{E}_1$ and $\mathbf{E}''_1 = (\mathbf{S}'')^{-1}\mathbf{E}_1$, and we solve the Y -problem in each of the two subspace collections $Y(j)$ and $Y(k)$, finding fields satisfying (7.5), (7.6), (7.8), and (7.9) with

$$\mathbf{J}'_2 = \mathbf{L}'\mathbf{E}_2, \quad \mathbf{J}''_2 = \mathbf{L}''\mathbf{E}''_2, \quad (7.11)$$

where

$$\mathbf{L}' = \sum_{i=1}^j z'_i \mathbf{\Lambda}'_i, \quad \mathbf{L}'' = \sum_{i=1}^k z''_i \mathbf{\Lambda}''_i, \quad (7.12)$$

and $\mathbf{\Lambda}'_i$ projects onto \mathcal{P}'_i while $\mathbf{\Lambda}''_i$ projects onto \mathcal{P}''_i . Hence we have

$$\mathbf{J}_2 = \mathbf{J}'_2 + \mathbf{J}''_2 = \mathbf{L}(\mathbf{E}'_2 + \mathbf{E}''_2), \quad \text{with } \mathbf{L} = \mathbf{L}' + \mathbf{L}''. \quad (7.13)$$

Then (7.10) implies

$$\mathbf{J}_1 = \mathbf{S}'\mathbf{J}'_1 + \mathbf{S}''\mathbf{J}''_1 = \mathbf{S}'\mathbf{Y}'\mathbf{E}'_1 + \mathbf{S}''\mathbf{Y}''\mathbf{E}''_1 = \mathbf{Y}\mathbf{E}_1, \quad (7.14)$$

where

$$\mathbf{Y} = \mathbf{S}'\mathbf{Y}'(\mathbf{S}')^{-1} + \mathbf{S}''\mathbf{Y}''(\mathbf{S}'')^{-1}. \quad (7.15)$$

If we have a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ for \mathcal{V} , then it is natural to take $(\mathbf{S}')^{-1}\mathbf{v}_1, (\mathbf{S}')^{-1}\mathbf{v}_2, \dots, (\mathbf{S}')^{-1}\mathbf{v}_n$ as a basis for \mathcal{V}' , and to take $(\mathbf{S}'')^{-1}\mathbf{v}_1, (\mathbf{S}'')^{-1}\mathbf{v}_2, \dots, (\mathbf{S}'')^{-1}\mathbf{v}_n$ as a basis for \mathcal{V}'' . Then the operators \mathbf{S}' and \mathbf{S}'' are represented by identity matrices, and in these bases (7.15) becomes $\mathbf{Y} = \mathbf{Y}' + \mathbf{Y}''$.

In the case where either or both of the subspaces \mathcal{V}' and \mathcal{V}'' have dimension less than the dimension n of the subspace \mathcal{V} we can first do an embedding. For example suppose \mathcal{V}' has dimension $n' < n$. Then let \mathcal{W}' be a space of dimension $n - n'$. Construct the subspace collection

$$\tilde{\mathcal{K}}' = \tilde{\mathcal{E}}' \oplus \mathcal{J}' = \tilde{\mathcal{V}}' \oplus \mathcal{H}', \quad (7.16)$$

where

$$\tilde{\mathcal{V}}' = \mathcal{V}' \oplus \mathcal{W}', \quad \tilde{\mathcal{E}}' = \mathcal{E}' \oplus \mathcal{W}'. \quad (7.17)$$

Then given a field $\tilde{\mathbf{E}}'_1 \in \tilde{\mathcal{V}}'$ we can express it as a sum $\mathbf{E}'_1 + \mathbf{W}'$ with $\mathbf{E}'_1 \in \mathcal{V}'$ and $\mathbf{W}' \in \mathcal{W}'$. We write $\mathbf{E}'_1 = \Psi\tilde{\mathbf{E}}'_1$ where Ψ is the projection onto \mathcal{V}' . Given this \mathbf{E}'_1 and solving the Y -problem associated with \mathcal{K}' we obtain fields \mathbf{E}' and \mathbf{J}' satisfying

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}'_1 + \mathbf{E}'_2 \in \mathcal{E}', \quad \mathbf{E}'_1 \in \mathcal{V}', \quad \mathbf{E}'_2 \in \mathcal{H}', \\ \mathbf{J}' &= \mathbf{J}'_1 + \mathbf{J}'_2 \in \mathcal{J}', \quad \mathbf{J}'_1 \in \mathcal{V}', \quad \mathbf{J}'_2 = \mathbf{L}\mathbf{E}_2 \in \mathcal{H}. \end{aligned} \quad (7.18)$$

It follows that the Y -problem in the space $\tilde{\mathcal{K}}'$ is solved with fields

$$\tilde{\mathbf{E}}' = \mathbf{W} + \mathbf{E}' = \mathbf{W} + \mathbf{E}'_1 + \mathbf{E}'_2, \quad \text{and} \quad \mathbf{J}' = \mathbf{J}'_1 + \mathbf{J}'_2 \quad \text{with} \quad \mathbf{J}'_2 = \mathbf{L}\mathbf{E}_2, \quad (7.19)$$

implying that

$$\mathbf{J}'_1 = -\mathbf{Y}\mathbf{E}'_1 = -\mathbf{Y}\Psi\tilde{\mathbf{E}}'_1. \quad (7.20)$$

We conclude that the new Y -problem has an operator $\tilde{\mathbf{Y}} = \mathbf{Y}\Psi$, i.e. its range is not the whole space $\tilde{\mathcal{V}}'$ but only at most the subspace \mathcal{V}' . After making such embeddings to ensure that \mathcal{V}' and \mathcal{V}'' (or rather $\tilde{\mathcal{V}}'$ and $\tilde{\mathcal{V}}''$ have the same dimension as the dimension n of the subspace \mathcal{V} , we are then free to add them.

The additive zero is easy to find. Let us consider the degenerate subspace collection

$$\mathcal{K}'' = \mathcal{E}'' = \mathcal{V}'' \quad (7.21)$$

Clearly \mathcal{H}'' contains only the zero vector, and we are forced to choose $\mathbf{L}'' = 0$. Given $\mathbf{E}_1 \in \mathcal{V}''$. The Y -problem is solved with vectors

$$\mathbf{E}'' = \mathbf{E}_1, \quad \mathbf{E}_1 = \mathbf{J}_1 = \mathbf{J}_2 = \mathbf{J} = 0. \quad (7.22)$$

Implying the associated Y -operator \mathbf{Y} is zero: thus the subspace collection (7.21) is the additive zero. Note that this subspace collection does not satisfy the property $\mathcal{E}'' \cap \mathcal{V}'' = 0$ which is needed for the inverse of \mathbf{Y} to exist, which is not surprising since $\mathbf{Y} = 0$ has no inverse.

Now suppose we have a subspace collection

$$\mathcal{K}' = \mathcal{E}' \oplus \mathcal{J}' = \mathcal{V}' \oplus \mathcal{H}' \quad \text{with} \quad \mathcal{H}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_j, \quad (7.23)$$

with associated operator $\mathbf{Y}'(z'_1, z'_2, \dots, z'_n)$ when

$$\mathbf{L}' = \sum_{i=1}^j z'_i \mathbf{\Lambda}'_i. \quad (7.24)$$

It is clear that if we replace \mathbf{L}' by

$$\mathbf{L}' = - \sum_{i=1}^j z'_i \mathbf{\Lambda}'_i, \quad (7.25)$$

then the solution to the Y -problem will give the \mathbf{Y} -operator

$$\mathbf{Y}(-z'_1, -z'_2, \dots, -z'_n) = -\mathbf{Y}(z'_1, z'_2, \dots, z'_n), \quad (7.26)$$

where to obtain this last identity we have used the homogeneity of the function. Since adding (7.26) to the associated operator $\mathbf{Y}'(z'_1, z'_2, \dots, z'_n)$ we started with gives zero, it is tempting to conclude that we have found the additive inverse. However the function (7.26) is not the \mathbf{Y} -operator valued function of z'_1, z'_2, \dots, z'_n associated with the subspace collection (7.23), whose definition does not allow us to choose \mathbf{L}' of the form (7.25). This is made more clear in the case where we have an orthogonal subspace collection since then the imaginary part of $(\mathbf{V}, \mathbf{Y}(z'_1, z'_2, \dots, z'_n)\mathbf{V})$ is generally positive when z'_1, z'_2, \dots, z'_n all have positive imaginary parts, and $-\mathbf{Y}(z'_1, z'_2, \dots, z'_n)$ then does not share this Herglotz property. So the additive inverse of an orthogonal subspace collection should typically not be an orthogonal subspace collection. We will find the proper additive inverse in section 12.

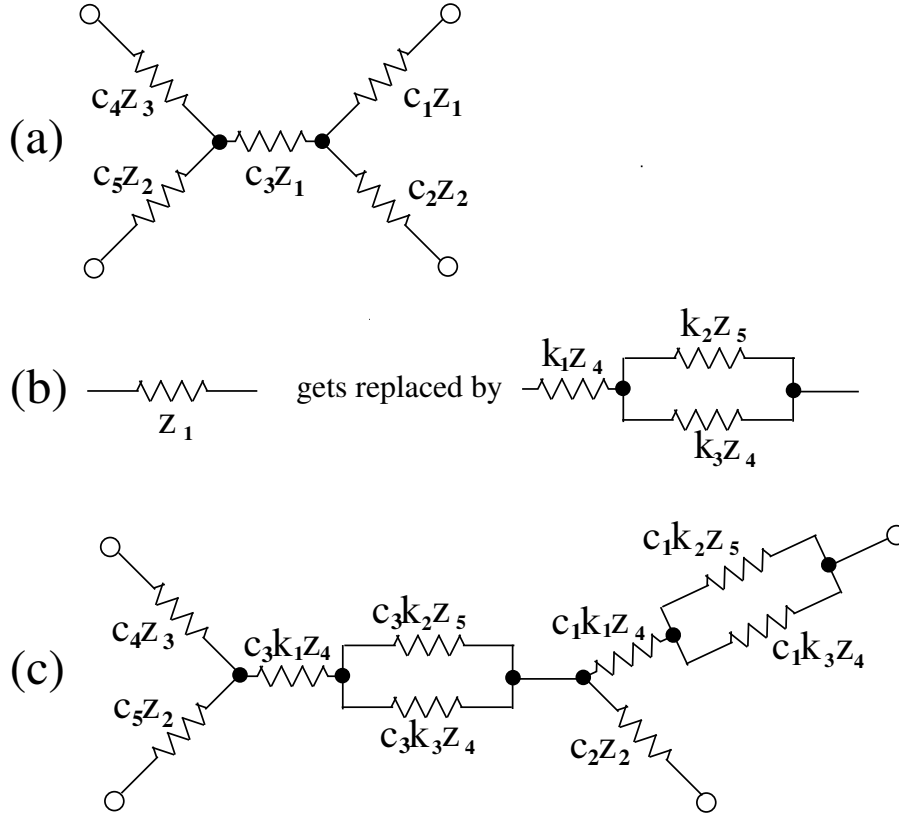


Figure 7: Substitution of Y - and Z -subspace collections is like replacing all resistors of one type by a compound network. If one takes a subspace collection, as, for example, represented by the 4-terminal network in (a) and replaces z_1 by the network in (b), where $k_1 + (1/k_2 + 1/k_3)^{-1} = 1$, to ensure this replacement does effect the resistance when $z_1 = z_4 = z_5 = 1$, one obtains the subspace collection as represented by the 4-terminal network in (c).

8 Substitution of subspace collections

Another familiar operation that we can do with rational functions is to make substitutions. Substitution of one subspace collection in another is similar to the way it can be done in electrical circuits. An example is shown in figure 7. Thus if $\mathbf{Y}(z_1, z_2, \dots, z_n)$ is a $m \times m$ matrix-valued homogeneous function of degree one and $Z'(z'_1, z'_2, \dots, z'_p)$ is a scalar-valued function, say normalized with

$$Z'(1, 1, \dots, 1) = 1, \quad (8.1)$$

then

$$\mathbf{Y}''(z'_1, z'_2, \dots, z'_p, z_2, \dots, z_n) = \mathbf{Y}(Z'(z'_1, z'_2, \dots, z'_p), z_2, \dots, z_n) \quad (8.2)$$

will be another $m \times m$ matrix-valued homogeneous function of degree one. What is the analogous operation on subspace collections? It is natural to expect there should be one, just as in a network of n types of resistors one can replace each resistor of type 1 with a network of p other resistors.

Extending the treatment given in Section 29.1 of Milton (2002) let us suppose that

we are given a $Y(n)$ -subspace collection

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (8.3)$$

and a $(3, p)$ -subspace collection

$$\mathcal{H}' = \mathcal{U}' \oplus \mathcal{E}' \oplus \mathcal{J}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_p, \quad (8.4)$$

in which \mathcal{V} is m -dimensional and \mathcal{U}' is one-dimensional. Let $\mathbf{Y}(z_1, z_2, \dots, z_n)$ and $Z'(z'_1, z'_2, \dots, z'_p)$ denote the functions associated with these subspace collections. We take as our new $(2, n+p)$ -subspace collection,

$$\mathcal{K}'' = \mathcal{E}'' \oplus \mathcal{J}'' = \mathcal{V}'' \oplus \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_n, \quad (8.5)$$

where

$$\mathcal{E}'' = (\mathcal{E} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{E}'), \quad \mathcal{J}'' = (\mathcal{J} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{J}'), \quad (8.6)$$

and

$$\begin{aligned} \mathcal{V}'' &= \mathcal{V} \otimes \mathcal{U}', \\ \mathcal{P}''_i &= \mathcal{P}_1 \otimes \mathcal{P}'_i \quad \text{for } 1 \leq i \leq p, \\ &= \mathcal{P}_{i+1-p} \otimes \mathcal{U}' \quad \text{for } p+1 \leq i \leq n+p-1. \end{aligned} \quad (8.7)$$

in which \otimes denotes the operation of taking the tensor product of two subspaces. Vectors in the space

$$\mathcal{K}'' = \mathcal{E}'' \oplus \mathcal{J}'' = (\mathcal{K} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes (\mathcal{E}' \oplus \mathcal{J}')) \quad (8.8)$$

spanned by these subspaces are represented as a pair $[\mathbf{P}, \mathbf{u}']$ added to a linear combination of pairs of the form $[\mathbf{P}_1, \mathbf{P}']$, where $\mathbf{P} \in \mathcal{K}$, $\mathbf{u}' \in \mathcal{U}'$, $\mathbf{P}_1 \in \mathcal{P}_1$, and $\mathbf{P}' \in \mathcal{E}' \oplus \mathcal{J}'$.

Now define

$$\begin{aligned} \mathcal{H} &= \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \\ \mathcal{H}'' &= \mathcal{P}''_1 \oplus \mathcal{P}''_2 \oplus \cdots \oplus \mathcal{P}''_n, \end{aligned} \quad (8.9)$$

and suppose that we are given solutions to the equations

$$\begin{aligned} \mathbf{J}_2 &= \sum_{i=1}^n z_i \mathbf{\Lambda}_i \mathbf{E}_2 \quad \text{with } \mathbf{E}_1 + \mathbf{E}_2 \in \mathcal{E}, \quad \mathbf{J}_1 + \mathbf{J}_2 \in \mathcal{J}, \quad \mathbf{E}_1, \mathbf{J}_1 \in \mathcal{V}, \quad \mathbf{E}_2, \mathbf{J}_2 \in \mathcal{H}, \\ \mathbf{j}' + \mathbf{J}' &= \sum_{j=1}^n z'_j \mathbf{\Lambda}'_j (\mathbf{e}' + \mathbf{E}') \quad \text{with } \mathbf{e}', \mathbf{j}' \in \mathcal{U}', \quad \mathbf{E}' \in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}', \end{aligned} \quad (8.10)$$

where

$$z_1 = Z(z'_1, z'_2, \dots, z'_p), \quad (8.11)$$

while $\mathbf{\Lambda}_i$ and $\mathbf{\Lambda}'_j$ are the projections onto \mathcal{P}_i and \mathcal{P}'_j . Let us introduce

$$\mathbf{P}_i = \mathbf{\Lambda}_i \mathbf{E}_2, \quad \mathbf{P}'_j = \mathbf{\Lambda}'_j (\mathbf{e}' + \mathbf{E}'), \quad (8.12)$$

and set

$$\begin{aligned} z_i'' &= z_i' \quad \text{for } 1 \leq i \leq p, \\ &= z_{i+1-p} \quad \text{for } p+1 \leq i \leq n+p-1. \end{aligned} \quad (8.13)$$

Then, in the new subspace collection, the vectors

$$\begin{aligned} \mathbf{E}_1'' &= [\mathbf{E}_1, \mathbf{e}'] \in \mathcal{V}'', \quad \mathbf{E}_2'' = [\mathbf{E}_2, \mathbf{e}'] + [\mathbf{P}_1, \mathbf{E}'], \\ \mathbf{J}_1'' &= [\mathbf{J}_1, \mathbf{e}'] \in \mathcal{V}'', \quad \mathbf{J}_2'' = [\mathbf{J}_2, \mathbf{e}'] + [\mathbf{P}_1, \mathbf{J}'] \end{aligned} \quad (8.14)$$

satisfy

$$\mathbf{E}_1'' + \mathbf{E}_2'' \in \mathcal{E}'', \quad \mathbf{J}_1'' + \mathbf{J}_2'' \in \mathcal{J}''. \quad (8.15)$$

Additionally, we have

$$\mathbf{E}_2'' = \left[\sum_{i=1}^n \mathbf{P}_i, \mathbf{e}' \right] - [\mathbf{P}_1, \mathbf{e}'] + [\mathbf{P}_1, \sum_{i=1}^p \mathbf{P}_i'] = \sum_{i=1}^{n+p-1} \mathbf{P}_i'' \in \mathcal{H}'', \quad (8.16)$$

where

$$\begin{aligned} \mathbf{P}_i'' &= [\mathbf{P}_1, \mathbf{P}_i'] \quad \text{for } 1 \leq i \leq p, \\ &= [\mathbf{P}_{i+1-p}, \mathbf{e}'] \quad \text{for } p+1 \leq i \leq n+p-1 \end{aligned} \quad (8.17)$$

satisfies $\mathbf{P}_i'' \in \mathcal{P}_i''$. Similarly, and using the fact implied by (8.11) that $\mathbf{j}' = Z\mathbf{e}' = z_1\mathbf{e}'$, we have

$$\mathbf{J}_2'' = \left[\sum_{i=1}^n z_i \mathbf{P}_i, \mathbf{e}' \right] - [\mathbf{P}_1, \mathbf{j}'] + [\mathbf{P}_1, \sum_{i=1}^p z_i' \mathbf{P}_i'] = \sum_{i=1}^{n+p-1} z_i'' \mathbf{P}_i'' \in \mathcal{H}''. \quad (8.18)$$

Given a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ for \mathcal{V} and a vector \mathbf{u}' in \mathcal{U}' it is natural to take $(\mathbf{v}_1, \mathbf{u}'), (\mathbf{v}_2, \mathbf{u}'), \dots, (\mathbf{v}_m, \mathbf{u}')$ as our basis for \mathcal{V}'' . Choosing \mathbf{e}' so that $\mathbf{e}' = \mathbf{u}'$, it is evident that $\mathbf{Y}(Z'(z_1', z_2', \dots, z_p'), z_2, \dots, z_n)$ is the matrix-valued function associated the new subspace collection, represented in these bases.

There is a similar subspace operation corresponding to substituting the Z -function $Z'(z_1', z_2', \dots, z_p')$ into another Z -function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ to obtain

$$\mathbf{Z}''(z_1', z_2', \dots, z_p', z_2, \dots, z_n) = \mathbf{Z}(Z(z_1', z_2', \dots, z_p'), z_2, \dots, z_n). \quad (8.19)$$

Given a $Z(n)$ -subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, \quad (8.20)$$

and a $(3, p)$ -subspace collection

$$\mathcal{H}' = \mathcal{U}' \oplus \mathcal{E}' \oplus \mathcal{J}' = \mathcal{P}_1' \oplus \mathcal{P}_2' \oplus \dots \oplus \mathcal{P}_p', \quad (8.21)$$

in which \mathcal{U} is m -dimensional and \mathcal{U}' is one-dimensional, we take as our new $(3, n+p-1)$ -subspace collection,

$$\mathcal{K}'' = \mathcal{U}'' \oplus \mathcal{E}'' \oplus \mathcal{J}'' = \mathcal{P}_1'' \oplus \mathcal{P}_2'' \oplus \dots \oplus \mathcal{P}_n'', \quad (8.22)$$

where

$$\mathcal{U}'' = \mathcal{U} \otimes \mathcal{U}', \quad \mathcal{E}'' = (\mathcal{E} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{E}'), \quad \mathcal{J}'' = (\mathcal{J} \otimes \mathcal{U}') \oplus (\mathcal{P}_1 \otimes \mathcal{J}'), \quad (8.23)$$

and

$$\begin{aligned} \mathcal{P}_i'' &= \mathcal{P}_1 \otimes \mathcal{P}_i' \quad \text{for } 1 \leq i \leq p, \\ &= \mathcal{P}_{i+1-p} \otimes \mathcal{U}' \quad \text{for } p+1 \leq i \leq n+p-1. \end{aligned} \quad (8.24)$$

Suppose that we are given solutions to the equations

$$\begin{aligned} \mathbf{j} + \mathbf{J} &= \sum_{i=1}^n z_i \mathbf{\Lambda}_i(\mathbf{e} + \mathbf{E}) \quad \text{with } \mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \\ \mathbf{j}' + \mathbf{J}' &= \sum_{j=1}^p z'_j \mathbf{\Lambda}'_j(\mathbf{e}' + \mathbf{E}') \quad \text{with } \mathbf{e}', \mathbf{j}' \in \mathcal{U}', \quad \mathbf{E}' \in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}', \end{aligned} \quad (8.25)$$

where $z_1 = Z(z'_1, z'_2, \dots, z'_p)$, while $\mathbf{\Lambda}_i$ and $\mathbf{\Lambda}'_j$ are the projections onto \mathcal{P}_i and \mathcal{P}'_j . Let us introduce

$$\mathbf{P}_i = \mathbf{\Lambda}_i(\mathbf{e} + \mathbf{E}), \quad \mathbf{P}'_j = \mathbf{\Lambda}'_j(\mathbf{e}' + \mathbf{E}'),$$

and define z''_i by (8.13), and $\mathbf{P}''_i \in \mathcal{P}''_i$ by (8.17). In the new subspace collection, the vectors

$$\begin{aligned} \mathbf{e}'' &= [\mathbf{e}, \mathbf{e}'] \in \mathcal{U}'', \quad \mathbf{E}'' = [\mathbf{E}, \mathbf{e}'] + [\mathbf{P}_1, \mathbf{E}'] \in \mathcal{E}'', \\ \mathbf{j}'' &= [\mathbf{j}, \mathbf{e}'] \in \mathcal{U}'', \quad \mathbf{J}'' = [\mathbf{J}, \mathbf{e}'] + [\mathbf{P}_1, \mathbf{J}'] \in \mathcal{J}'' \end{aligned} \quad (8.26)$$

satisfy

$$\begin{aligned} \mathbf{e}'' + \mathbf{E}'' &= \left[\sum_{i=1}^n \mathbf{P}_i, \mathbf{e}' \right] + [\mathbf{P}_1, \sum_{j=1}^p \mathbf{P}'_j] - [\mathbf{P}_1, \mathbf{e}'] \\ &= \left[\sum_{i=2}^n \mathbf{P}_i, \mathbf{e}' \right] + [\mathbf{P}_1, \sum_{j=1}^p \mathbf{P}'_j] \\ &= \sum_{i=1}^{n+p-1} \mathbf{P}''_i, \end{aligned} \quad (8.27)$$

and, using (8.11),

$$\begin{aligned} \mathbf{j}'' + \mathbf{J}'' &= \left[\sum_{i=1}^n z_i \mathbf{P}_i, \mathbf{e}' \right] + [\mathbf{P}_1, \sum_{j=1}^p z'_j \mathbf{P}'_j] - [\mathbf{P}_1, \mathbf{j}'] \\ &= \left[\sum_{i=2}^n z_i \mathbf{P}_i, \mathbf{e}' \right] + [\mathbf{P}_1, \sum_{j=1}^p z'_j \mathbf{P}'_j] \\ &= \sum_{i=1}^{n+p-1} z''_i \mathbf{P}''_i. \end{aligned} \quad (8.28)$$

Given a basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for \mathcal{U} and a vector \mathbf{u}' in \mathcal{U}' it is natural to take $(\mathbf{u}_1, \mathbf{u}'), (\mathbf{u}_2, \mathbf{u}'), \dots, (\mathbf{u}_m, \mathbf{u}')$ as our basis for \mathcal{U}'' . Choosing \mathbf{e}' so that $\mathbf{e}' = \mathbf{u}'$, it is evident from (8.26) that $\mathbf{Z}(Z'(z'_1, z'_2, \dots, z'_p), z_2, \dots, z_n)$ is the matrix-valued function associated the new subspace collection, represented in these bases.

9 Some other elementary operations on subspace collections

A further operation we can do on functions $\mathbf{Y}(z_1, z_2, \dots, z_n)$ while retaining the homogeneity of degree 1 in the variables z_1, z_2, \dots, z_n is to replace the function by $[\mathbf{Y}(1/z_1, 1/z_2, \dots, 1/z_n)]^{-1}$. The analogous operation on the associated $Y(n)$ -subspace collection is to interchange the subspaces \mathcal{E} and \mathcal{J} . Similarly in a $Z(n)$ subspace collection, interchanging the subspaces \mathcal{E} and \mathcal{J} corresponds to replacing $\mathbf{Z}(z_1, z_2, \dots, z_n)$ by $[\mathbf{Z}(1/z_1, 1/z_2, \dots, 1/z_n)]^{-1}$, as noted in Section 29.1 of Milton (2002). We call such a transformation a duality transformation. As a consequence of the duality transformation (4.3) immediately implies the formula

$$\mathbf{Z}^{-1} = \Gamma_0[(\Gamma_0 + \Gamma_1)\mathbf{L}(\Gamma_0 + \Gamma_1)]^{-1}\Gamma_0. \quad (9.1)$$

One simple thing we can do in a function is set $z_j = z_k$: the analogous operation in a subspace collection is to replace $\mathcal{P}_j \oplus \mathcal{P}_k$ by a single subspace.

Another elementary operation we can do on a $Z(n)$ subspace collection is as follows. Let \mathcal{U} be expressed as the direct sum

$$\mathcal{U} = \mathcal{U}' \oplus \mathcal{W}, \quad (9.2)$$

which defines the projection Φ onto \mathcal{U}' . We now take as our subspace collection

$$\mathcal{H} = \mathcal{U}' \oplus \mathcal{E} \oplus \mathcal{J}' = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, \quad (9.3)$$

where

$$\mathcal{J}' = \mathcal{J} \oplus \mathcal{W}. \quad (9.4)$$

Then any solution to the Z -problem (2.23) with $\mathbf{e} \in \mathcal{U}'$ immediately generates a solution to the Z -problem associated with the subspace collection (9.3):

$$\mathbf{j}' \in \mathcal{U}', \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J}' \in \mathcal{J}', \quad \mathbf{j}' + \mathbf{J}' = \mathbf{L}(\mathbf{e} + \mathbf{E}), \quad (9.5)$$

where

$$\mathbf{L} = \sum_{i=1}^n z_i \mathbf{\Lambda}_i, \quad \mathbf{j}' = \Phi \mathbf{j}, \quad \mathbf{J}' = \mathbf{J} + (\mathbf{I} - \Phi)\mathbf{j}, \quad (9.6)$$

which ensures that

$$\mathbf{j} + \mathbf{J} = \mathbf{j}' + \mathbf{J}' \quad \text{and} \quad (\mathbf{I} - \Phi)\mathbf{j} \in \mathcal{W}. \quad (9.7)$$

Hence the new subspace collection has a \mathbf{Z} -operator

$$\mathbf{Z}' = \Phi \mathbf{Z}, \quad (9.8)$$

when applied to fields in \mathcal{U}' .

10 Realizing any Y -matrix with elements that are rational functions of degree 1

Given any homogeneous rational function of degree 1,

$$Z(z_1, z_2, \dots, z_n) = \frac{p(z_1, z_2, \dots, z_n)}{q(z_1, z_2, \dots, z_n)}, \quad (10.1)$$

satisfying the normalization $Z(1, 1, \dots, 1) = 1$ where $p(z_1, z_2, \dots, z_n)$ and $q(z_1, z_2, \dots, z_n)$ are homogeneous polynomials of degree k and $k - 1$ respectively, where k is a positive integer, our first goal is to find a $Z(n)$ subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, \quad (10.2)$$

where \mathcal{U} is one-dimensional which has $Z(z_1, z_2, \dots, z_n)$ as its associated function. Without loss of generality we could set $z_n = 1$, and then $p(z_1, z_2, \dots, z_{n-1}, 1)$ and $q(z_1, z_2, \dots, z_{n-1}, 1)$ are just arbitrary polynomials of the $n - 1$ variables z_1, z_2, \dots, z_{n-1} . Also without loss of generality we can assume

$$p(1, 1, \dots, 1) = q(1, 1, \dots, 1) = 1. \quad (10.3)$$

The first step is to realize $Z(z_1, z_2, 1) = z_1 z_2$ as an associated Z -function. Note that (3.25) implies we can realize

$$Z(z_1, 1) = z_1^2, \quad (10.4)$$

and (3.26) implies we can realize

$$Z(z_1, z_2) = cz_1 + (1 - c)z_2, \quad (10.5)$$

for any constant c . Hence, by substitution we can realize

$$Z(z_1, z_2, 1) = 9(2z_1/3 + z_2/3)^2/8 - (2z_1 - z_2)^2/8 = z_1 z_2. \quad (10.6)$$

Making further substitutions, we can realize any product of the variables

$$Z(z_1, z_2, \dots, z_{n-1}, 1) = z_1^{a_1} z_2^{a_2} \dots z_{n-1}^{a_{n-1}}, \quad (10.7)$$

where the a_i are nonnegative integers. By repeated substitution in (10.5) we can realize any linear combination of such terms with coefficients summing to 1. Thus we can realize the polynomials $p(z_1, z_2, \dots, z_{n-1}, 1)$ and $q(z_1, z_2, \dots, z_{n-1}, 1)$.

Furthermore (3.25), with the roles of z_1 and z_2 interchanged, implies we can realize

$$Z(z_1, 1) = 1/z_1, \quad (10.8)$$

which by substitution into (10.6) implies we can realize

$$Z(z_1, z_2, 1) = z_2/z_1. \quad (10.9)$$

Substituting $p(z_1, z_2, \dots, z_{n-1}, 1)$ for z_2 and $q(z_1, z_2, \dots, z_{n-1}, 1)$ for z_1 we see we can find a subspace collection which realizes

$$Z(z_1, z_2, \dots, z_{n-1}, 1) = \frac{p(z_1, z_2, \dots, z_{n-1}, 1)}{q(z_1, z_2, \dots, z_{n-1}, 1)} \quad (10.10)$$

as its associated Z -function when $z_n = 1$. When z_n is not 1, the subspace collection will by homogeneity realize the function (10.1).

Now from (3.13) we can realize

$$\mathbf{Y}(z_1) = \begin{pmatrix} a_{11}z_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (10.11)$$

and realize

$$\mathbf{Y}(z_2) = \begin{pmatrix} 0 & a_{12}z_2 \\ 0 & 0 \end{pmatrix}. \quad (10.12)$$

By substitution of subspace collections, we can realize any Y -matrix where in the above formulae z_1 and z_2 are replaced by any normalized rational homogeneous functions of degree 1 (normalized in the sense that they take the value 1 when all variables take the value 1). Finally, by making suitable embeddings and adding subspace collections we can realize any Y -matrix with elements that are homogeneous rational functions of degree 1: (10.11) with the appropriate substitutions realizes each diagonal element, while (10.12) with the appropriate substitutions realizes each off-diagonal element.

11 Extension operations on subspace collections

Let us suppose we have a $Z(n)$ subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (11.1)$$

where \mathcal{U} is m -dimensional. Let \mathcal{V} be another m -dimensional space, and consider the space

$$\mathcal{K} = \mathcal{V} \oplus \mathcal{H}. \quad (11.2)$$

Suppose there is a nonsingular mapping \mathbf{T} from \mathcal{U} to \mathcal{V} . Define the subspace $\tilde{\mathcal{E}}$ to consist of all vectors spanned by $\mathbf{u} + \mathbf{T}\mathbf{u}$ as \mathbf{u} varies in \mathcal{U} . Define $\tilde{\mathcal{J}}$ to consist of all vectors spanned by $\mathbf{u} - \mathbf{T}\mathbf{u}$ as \mathbf{u} varies in \mathcal{U} . Clearly we have

$$\mathcal{V} \oplus \mathcal{U} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{J}}, \quad (11.3)$$

and consequently we obtain the $Y(n)$ subspace collection

$$\mathcal{K} = \mathcal{E}' \oplus \mathcal{J}' = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n, \quad (11.4)$$

in which

$$\mathcal{E}' = \tilde{\mathcal{E}} \oplus \mathcal{E}, \quad \mathcal{J}' = \tilde{\mathcal{J}} \oplus \mathcal{J}. \quad (11.5)$$

Furthermore given vectors satisfying

$$\mathbf{j} + \mathbf{J} = \mathbf{L}(\mathbf{e} + \mathbf{E}), \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad (11.6)$$

where

$$\mathbf{j} = \mathbf{Z}\mathbf{e}, \quad \mathbf{L} = \sum_{\ell=1}^n z_{\ell} \mathbf{\Lambda}_{\ell}, \quad (11.7)$$

we can set

$$\mathbf{E}_2 = \mathbf{e} + \mathbf{E} \in \mathcal{H}, \quad \mathbf{E}_1 = \mathbf{T}\mathbf{e} \in \mathcal{V}, \quad \mathbf{J}_2 = \mathbf{j} + \mathbf{J} \in \mathcal{H}, \quad \mathbf{J}_1 = -\mathbf{T}\mathbf{j}. \quad (11.8)$$

Then we have

$$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{T}\mathbf{e} + \mathbf{e} + \mathbf{E} \in \mathcal{E}', \quad \mathbf{J}_1 + \mathbf{J}_2 = -\mathbf{T}\mathbf{j} + \mathbf{j} + \mathbf{J} \in \mathcal{J}', \quad (11.9)$$

and

$$\mathbf{J}_1 = -\mathbf{Y}\mathbf{E}_1, \quad \text{with } \mathbf{Y} = \mathbf{T}\mathbf{Z}\mathbf{T}^{-1}. \quad (11.10)$$

Given a basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for \mathcal{U} , with respect to which the matrix valued function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ is defined, it is natural to take $\mathbf{T}\mathbf{u}_1, \mathbf{T}\mathbf{u}_2, \dots, \mathbf{T}\mathbf{u}_m$ as our basis for \mathcal{V} . Then \mathbf{T} is represented by the identity matrix, and the functions $\mathbf{Z}(z_1, z_2, \dots, z_n)$ and $\mathbf{Y}(z_1, z_2, \dots, z_n)$ are identical. We call the subspace collection (11.4) the extension of the subspace collection (11.1).

12 Reference transformations and additive inverses

Given the impedance network illustrated in Figure 3 we are free to change the scaling constants c_i assigned to each bond to new constants c'_i and accordingly replace z_i with $z'_i = z_i c_i / c'_i$ without changing the overall electrical response of the network. Analogously, given a homogeneous rational function $\mathbf{Y}(z_1, z_2, \dots, z_n)$ of degree one, an operation which preserves the homogeneity is obviously to multiply the variables by constants to obtain the function

$$\mathbf{Y}'(z'_1, z'_2, \dots, z'_n) = \mathbf{Y}(d_1 z'_1, d_2 z'_2, \dots, d_n z'_n). \quad (12.1)$$

The associated operation on the $Y(n)$ subspace collection $(\mathcal{E}, \mathcal{J})$ and $(\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ is found by generalizing the analysis given after (29.3) in Milton (2002) and is as follows. Given nonzero (possibly complex) constants c_i^E and c_i^J , $i = 1, \dots, n$ we introduce the linear transformations

$$\psi^E(\mathbf{P}) = \Pi_1 \mathbf{P} + \sum_{i=1}^n c_i^E \Lambda_i \mathbf{P}, \quad \psi^J(\mathbf{P}) = \Pi_1 \mathbf{P} + \sum_{i=1}^n c_i^J \Lambda_i \mathbf{P}, \quad (12.2)$$

on fields $\mathbf{P} \in \mathcal{K}$, where Λ_1 is the projection onto \mathcal{P}_1 . These transformations leave the subspaces \mathcal{V} and \mathcal{P}_i invariant. Define the spaces

$$\mathcal{E}' = \psi^E(\mathcal{E}) \quad \text{and} \quad \mathcal{J}' = \psi^J(\mathcal{J}). \quad (12.3)$$

These will have the same dimension as \mathcal{E} and \mathcal{J} respectively. To see this, suppose $\psi^E(\mathbf{E}) = \psi^E(\mathbf{E}')$ for some $\mathbf{E}, \mathbf{E}' \in \mathcal{E}$. Then $\psi^E(\mathbf{E} - \mathbf{E}') = 0$ and since (12.2) implies $\psi^E(\mathbf{P}) = 0$ only when $\mathbf{P} = 0$ we conclude that $\mathbf{E} = \mathbf{E}'$. We need to make the technical assumption that

$$\psi^E(\mathbf{E}) \neq \psi^J(\mathbf{J}), \quad \text{for all nonzero } \mathbf{E} \in \mathcal{E}, \mathbf{J} \in \mathcal{J}, \quad (12.4)$$

to ensure \mathcal{E}' and \mathcal{J}' have no nonzero vector in common. A more insightful meaning to the condition (12.4) is given in the next section.

Let $(\mathcal{E}', \mathcal{J}')$ and $(\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ be our new subspace collection. Given a solution to the equations

$$\mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad (\mathbf{I} - \Pi_1)\mathbf{J} = \sum_{i=1}^n z_i \Lambda_i \mathbf{E}, \quad (12.5)$$

in the original subspace collection, in which Π_1 is the projection onto \mathcal{V} , the fields $\mathbf{E}' = \psi^E(\mathbf{E})$ and $\mathbf{J}' = \psi^J(\mathbf{J})$ will be a solution to the equations

$$\mathbf{E}' \in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}', \quad (\mathbf{I} - \Pi_1)\mathbf{J}' = \sum_{i=1}^n z'_i \Lambda_i \mathbf{E}', \quad (12.6)$$

in the new subspace collection with

$$z'_i = z_i c_i^J / c_i^E. \quad (12.7)$$

Since $\Pi_1 \mathbf{E}' = \Pi_1 \mathbf{E}$ and $\Pi_1 \mathbf{J}' = \Pi_1 \mathbf{J}$, it follows that the \mathbf{Y} -tensor functions of the two subspace collections are related by (12.1) where

$$d_i = c_i^E / c_i^J. \quad (12.8)$$

In particular, if we choose $c_i^E = -c_i^J$ for all i we obtain $d_i = -1$. Then using the homogeneity of the function $\mathbf{Y}(z_1, z_2, \dots, z_n)$ we see that

$$\mathbf{Y}'(z'_1, z'_2, \dots, z'_n) = \mathbf{Y}(-z'_1, -z'_2, \dots, -z'_n) = -\mathbf{Y}(z'_1, z'_2, \dots, z'_n). \quad (12.9)$$

So if to another subspace collection, with an associated function $\mathbf{Y}''(z_1, z_2, \dots, z_n)$, we add this new subspace collection according to the prescription given in Section 7, then it produces a subspace collection with an associated Y -function which is obtained by subtracting $\mathbf{Y}(z_1, z_2, \dots, z_n)$ from $\mathbf{Y}''(z_1, z_2, \dots, z_n)$. In other words, when $c_i^E = -c_i^J$ for all i , the subspace collection with subspaces $(\mathcal{E}', \mathcal{J}')$ and $(\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ is the additive inverse of the original subspace collection, having subspaces $(\mathcal{E}, \mathcal{J})$ and $(\mathcal{V}, \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$, where $(\mathcal{E}', \mathcal{J}')$ and $(\mathcal{E}, \mathcal{J})$ are linked by (12.3). If the technical condition (12.4) is not satisfied it appears that the subspace collection has no arithmetic inverse.

13 Operations on subspace collections leaving the associated function invariant

Note from (12.8) that if we choose $c_i^J = c_i^E$ for all i then the associated function remains invariant. More generally, if we are interested in leaving the associated function invariant, we could take

$$\mathcal{E}' = \mathbf{C}\mathcal{E}, \quad \mathcal{J}' = \mathbf{C}\mathcal{J}, \quad \mathcal{V}' = \mathbf{C}\mathcal{V}, \quad \mathcal{H}' = \mathbf{C}\mathcal{H}, \quad \mathcal{P}'_i = \mathbf{C}\mathcal{P}_i, \quad (13.1)$$

where \mathbf{C} is a nonsingular linear operator which maps \mathcal{K} to itself. Then the fields $\mathbf{E}' = \mathbf{C}\mathbf{E}$ and $\mathbf{J}' = \mathbf{C}\mathbf{J}$ will be a solution to the equations

$$\mathbf{E}' \in \mathcal{E}', \quad \mathbf{J}' \in \mathcal{J}', \quad (\mathbf{I} - \Pi'_1)\mathbf{J}' = \sum_{i=1}^n z_i \Lambda'_i \mathbf{E}', \quad (13.2)$$

where

$$\Pi'_1 = \mathbf{C}\Pi_1\mathbf{C}^{-1}, \quad \Lambda'_i = \mathbf{C}\Lambda_i\mathbf{C}^{-1} \quad (13.3)$$

are the projections onto \mathcal{V}' and \mathcal{P}'_i . If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ is a basis for \mathcal{V} then setting $\mathbf{v}'_i = \mathbf{C}\mathbf{v}_i$ we can take $\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_m$ as a basis for \mathcal{V}' . Since multiplying by \mathbf{C} is a linear operation the coefficients in the expansions

$$\Pi'_1 \mathbf{E}' = \sum_{i=1}^m E'_i \mathbf{u}'_i, \quad \Pi_1 \mathbf{E} = \sum_{i=1}^m E_i \mathbf{u}_i, \quad \Pi'_1 \mathbf{J}' = \sum_{i=1}^m J'_i \mathbf{u}'_i, \quad \Pi_1 \mathbf{J} = \sum_{i=1}^m J_i \mathbf{u}_i \quad (13.4)$$

can be equated:

$$E'_i = E_i, \quad J'_i = J_i, \quad (13.5)$$

and as a consequence the same matrix \mathbf{Y} whose coefficients govern the relation

$$J_i = \sum_{k=1}^k Y_{ik} E_k, \quad (13.6)$$

also govern the relation

$$J'_i = \sum_{k=1}^k Y_{ik} E'_k. \quad (13.7)$$

Due to this equivalence it suffices in the preceeding section to limit attention to the transformations (12.2) having $c_i^J = 1$ for all i : it is only the ratio $d_i = c_i^E/c_i^J$ that has any real significance. Then $\psi^J(\mathbf{P}) = \mathbf{P}$, and the technical condition (12.4) is violated only when there are nonzero vectors $\mathbf{E} \in \mathcal{E}$ and $\mathbf{J} \in \mathcal{J}$ such that

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2, \quad \mathbf{E}_1 = \mathbf{J}_1 \in \mathcal{V}, \quad \mathbf{J}_2 = \mathbf{L}\mathbf{E}_2 \in \mathcal{H}, \quad \text{with } \mathbf{L} = \sum_{i=1}^n c_i^E \mathbf{A}_i. \quad (13.8)$$

Thus either $\mathbf{Y}(c_1^E, c_2^E, \dots, c_n^E)$ has an eigenvalue of -1 , or the Y -problem with $z_i = c_i^E$ for all i has a nonunique solution (with a nontrivial solution having $\mathbf{E}_2 \neq 0$ for the homogeneous problem with $\mathbf{E}_1 = 0$ and also $\mathbf{J}_1 = 0$, the latter not being needed for nonuniqueness but being needed if (13.8) holds). If we are looking for the arithmetic inverse we take $c_i^E = -1$ for all i , and the inverse exists except when $\mathbf{Y}(-1, -1, \dots, -1) = -\mathbf{Y}(1, 1, \dots, 1)$ has eigenvalue -1 or when the Y -problem with $z_i = 1$ for all i has a nonunique solution (with the homogeneous problem having a nontrivial solution with both \mathbf{E}_1 and \mathbf{J}_1 being zero).

There is a similar invariance of matrix functions associated with $Z(n)$ subspace collections under the linear transformations,

$$\mathcal{U}' = \mathbf{C}\mathcal{U}, \quad \mathcal{E}' = \mathbf{C}\mathcal{E}, \quad \mathcal{J}' = \mathbf{C}\mathcal{J}, \quad \mathcal{P}'_i = \mathbf{C}\mathcal{P}_i. \quad (13.9)$$

These invariances are quite natural, as they are isomorphic to changing the basis in the vector spaces \mathcal{H} or \mathcal{K} . Thus, up to these trivial equivalences, the arithmetic inverse defined in the previous section is unique.

14 Multiplicative Inverses of superfunctions

To find the multiplicative inverse of a superfunction $(F^s)'$ we let \mathcal{K}'' be a vector space with the same dimension as \mathcal{K}' , and we take \mathbf{C} as a nonsingular map from \mathcal{K}' to \mathcal{K}'' . We then let

$$\begin{aligned} \mathcal{J}'' &= \mathbf{C}(\mathcal{J}'), \quad \mathcal{H}'' = \mathbf{C}\mathcal{H}', \\ (\mathcal{V}^I)'' &= \mathbf{C}(\mathcal{V}^O)', \quad (\mathcal{V}^O)'' = \mathbf{C}(\mathcal{V}^I)', \quad \mathcal{P}''_i = \mathbf{C}\mathcal{P}'_i \text{ for } i = 1, 2, \dots, j. \end{aligned} \quad (14.1)$$

Introduce the transformation

$$\psi(\mathbf{P}) = \mathbf{\Pi}_1'' \mathbf{P} - \mathbf{\Pi}_2'', \quad (14.2)$$

where Π_1'' is the projection onto $(\mathcal{V}^I)'' \oplus (\mathcal{V}^O)''$ and Π_2'' is the projection onto \mathcal{H}'' . This is a special case of the transformations in (12.2). We let $\mathcal{E}'' = \psi(\mathbf{C}\mathcal{E}')$. Note that the output space $(\mathcal{V}^O)'$ gets mapped to the input space $(\mathcal{V}^I)''$, and the input space $(\mathcal{V}^I)'$ gets mapped to the output space $(\mathcal{V}^O)''$, and apart from these switchings we have essentially made an additive inverse in the Y -problem. We still require the technical condition mentioned in the last section, to ensure that this additive inverse exists: the operator $\mathbf{Y}(1, 1, \dots, 1) - \mathbf{I}$ is nonsingular and the Y -problem with $z_i = 1$ for all i has a unique solution (or more precisely the homogeneous problem does not have a nontrivial solution with both \mathbf{E}_1 and \mathbf{J}_1 being zero).

Now suppose we are given a solution to the superfunction problem associated with $(F^s)'$,

$$\mathbf{E}' = (\mathbf{E}^I)' + (\mathbf{E}^O)' + \mathbf{E}_2' \in \mathcal{E}', \quad \mathbf{J}' = (\mathbf{J}^I)' + (\mathbf{J}^O)' + \mathbf{J}_2' \in \mathcal{J}', \quad \mathbf{J}_2' = \mathbf{L}'\mathbf{E}_2', \quad (14.3)$$

where

$$\mathbf{L}' = \sum_{i=1}^j z_i' \mathbf{\Lambda}_i', \quad (14.4)$$

in which $\mathbf{\Lambda}_i'$ is the projection onto \mathcal{P}_i' , and

$$(\mathbf{E}^I)', (\mathbf{J}^I)' \in (\mathcal{V}^I)', \quad (\mathbf{E}^O)', (\mathbf{J}^O)' \in (\mathcal{V}^O)', \quad \mathbf{E}_2', \mathbf{J}_2' \in \mathcal{H}'. \quad (14.5)$$

Now take vectors

$$\begin{aligned} \mathbf{E}'' &= \psi(\mathbf{C}\mathbf{E}'), \quad \mathbf{J}'' = -\mathbf{C}\mathbf{J}', \quad \mathbf{E}_2'' = -\mathbf{C}\mathbf{E}_2', \quad \mathbf{J}_2'' = -\mathbf{C}\mathbf{J}_2' \\ (\mathbf{E}^I)'' &= \mathbf{C}(\mathbf{E}^O)', \quad (\mathbf{E}^O)'' = \mathbf{C}(\mathbf{E}^I)', \quad (\mathbf{J}^I)'' = -\mathbf{C}(\mathbf{J}^O)', \quad (\mathbf{J}^O)'' = -\mathbf{C}(\mathbf{J}^I)'. \end{aligned} \quad (14.6)$$

These solve the superfunction problem associated with $(F^s)''$,

$$\mathbf{E}'' = (\mathbf{E}^I)'' + (\mathbf{E}^O)'' + \mathbf{E}_2'' \in \mathcal{E}'', \quad \mathbf{J}'' = (\mathbf{J}^I)'' + (\mathbf{J}^O)'' + \mathbf{J}_2'' \in \mathcal{J}'', \quad \mathbf{J}_2'' = \mathbf{L}''\mathbf{E}_2'', \quad (14.7)$$

where

$$\mathbf{L}'' = \sum_{i=1}^j z_i'' \mathbf{\Lambda}_i'', \quad z_i'' = z_i', \quad (14.8)$$

in which $\mathbf{\Lambda}_i''$ is the projection onto \mathcal{P}_i'' , and

$$(\mathbf{E}^I)'', (\mathbf{J}^I)'' \in (\mathcal{V}^I)'', \quad (\mathbf{E}^O)'', (\mathbf{J}^O)'' \in (\mathcal{V}^O)'', \quad \mathbf{E}_2'', \mathbf{J}_2'' \in \mathcal{H}''. \quad (14.9)$$

Next let \mathbf{M}_1 denote the restriction of \mathbf{C} to the subspace $(\mathcal{V}^O)'$, i.e., that operator mapping $(\mathcal{V}^O)'$ to $(\mathcal{V}^I)''$, such that $\mathbf{M}_1\mathbf{P} = \mathbf{C}\mathbf{P}$ for all $\mathbf{P} \in (\mathcal{V}^O)'$. Then from (14.6) we have $(\mathbf{E}^I)'' = \mathbf{M}_1(\mathbf{E}^O)'$ and $(\mathbf{J}^I)'' = -\mathbf{M}_1(\mathbf{J}^O)'$. To see that $(F^s)''$ is the inverse of the superfunction $(F^s)'$ when

$$\mathbf{L}' = \sum_{i=1}^j z_i' \mathbf{\Lambda}_i', \quad \mathbf{L}'' = \sum_{i=1}^j z_i' \mathbf{\Lambda}_i'', \quad (14.10)$$

we introduce the operator \mathbf{M}_2 which is the restriction of \mathbf{C}^{-1} to the subspace $(\mathcal{V}^O)''$, i.e., that operator mapping $(\mathcal{V}^O)''$ to $(\mathcal{V}^I)'$, such that $\mathbf{M}_2\mathbf{P} = \mathbf{C}\mathbf{P}$ for all $\mathbf{P} \in (\mathcal{V}^I)'$. Then upon taking the product of the superfunctions (14.6) implies

$$\begin{pmatrix} (\mathbf{E}^O)'' \\ (\mathbf{J}^O)'' \end{pmatrix} = \mathbf{F} \begin{pmatrix} (\mathbf{E}^I)' \\ (\mathbf{J}^I)' \end{pmatrix}, \quad (14.11)$$

where

$$\mathbf{F} = \begin{pmatrix} (\mathbf{M}_2)^{-1} & 0 \\ 0 & -\mathbf{M}_2^{-1} \end{pmatrix} \quad (14.12)$$

is the multiplicative identity operator. From this analysis it looks like there are many multiplicative inverses, parameterized by \mathbf{C} , but in fact all are equivalent: this follows from the previous section.

15 Pruning the subspace collections

If an m terminal resistor network has a cluster of resistors which is not connected to the rest of the network, and that cluster does not have any terminals, only internal nodes, then clearly we can discard it without affecting the fields in the rest of the network and its response matrix. The analogous operation on subspace collections is called pruning.

When \mathbf{L} is close to $z_0\mathbf{I}$ we can expand the inverses in (4.5) and (4.7) to obtain the series expansions

$$\mathbf{E} = \sum_{j=1}^{\infty} [\mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})/z_0]^j \mathbf{e}, \quad (15.1)$$

$$\mathbf{Z} = z_0\mathbf{\Gamma}_0 + \sum_{j=1}^{\infty} \mathbf{\Gamma}_0(\mathbf{L} - z_0\mathbf{I})[\mathbf{\Gamma}_1(\mathbf{L} - z_0\mathbf{I})/z_0]^j \mathbf{\Gamma}_0. \quad (15.2)$$

From these expansions it is evident that it is only those fields in \mathcal{H} that arise from products of the operators $\mathbf{\Gamma}_1, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \dots, \mathbf{\Lambda}_n$ applied to fields in \mathcal{U} have any role in determining \mathbf{E} and the associated function $\mathbf{Z}(z_1, z_2, \dots, z_n)$ (also \mathbf{j} and \mathbf{J}): so we may as well prune away any other fields from the vector space \mathcal{H} . Thus we can redefine \mathcal{H} as the smallest subspace containing \mathcal{U} that is closed under the action of $\mathbf{\Gamma}_1, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \dots, \mathbf{\Lambda}_n$ and redefine

$$\mathcal{E} = \mathbf{\Gamma}_1\mathcal{H}, \quad \mathcal{J} = \mathbf{\Gamma}_2\mathcal{H}, \quad \mathcal{P}_j = \mathbf{\Lambda}_j\mathcal{H}, \quad j = 1, 2, \dots, n. \quad (15.3)$$

This imposes constraints on the dimensions of these subspaces, as noted in Section 29.2 of Milton (2002) where the results are given in the case where \mathcal{U} has dimension 1 and where the spaces are orthogonal. Let p_j be the dimension of \mathcal{P}_j , $j = 1, 2, \dots, n$, and let m, q_1 and q_2 represent the dimensions of \mathcal{U}, \mathcal{E} and \mathcal{J} . The total dimension of the vector space \mathcal{H} is therefore

$$h = m + q_1 + q_2 = p_1 + p_2 + \dots + p_n. \quad (15.4)$$

Now the space

$$[\mathbf{\Lambda}_1(\mathcal{U} \oplus \mathcal{E})] \oplus [\mathbf{\Lambda}_2(\mathcal{U} \oplus \mathcal{E})] \oplus \dots \oplus [\mathbf{\Lambda}_n(\mathcal{U} \oplus \mathcal{E})] \quad (15.5)$$

certainly contains \mathcal{U} , and is closed under $\mathbf{\Gamma}_1$ (because it contains \mathcal{E}) and is closed under $\mathbf{\Lambda}_j$ for each j . It therefore must be \mathcal{H} and $\mathbf{\Lambda}_j(\mathcal{U} \oplus \mathcal{E})$ which has at most dimension $m + q_1$ must be \mathcal{P}_j . Therefore for each j we have the inequality

$$p_j \leq m + q_1, \quad (15.6)$$

and by summing these over j we see that

$$q_2 \leq (n - 1)(m + q_1). \quad (15.7)$$

Similarly the subspace

$$[\mathbf{\Lambda}_1(\mathcal{U} \oplus \mathcal{J})] \oplus [\mathbf{\Lambda}_2(\mathcal{U} \oplus \mathcal{J})] \oplus \dots \oplus [\mathbf{\Lambda}_n(\mathcal{U} \oplus \mathcal{J})] \quad (15.8)$$

can also be identified with \mathcal{H} and we obtain the inequalities

$$p_j \leq m + q_2, \quad q_1 \leq (n-1)(m + q_2). \quad (15.9)$$

In the particular case when $n = 2$ the constraints (15.7) and (15.9) imply that the dimensions of the subspaces \mathcal{E} and \mathcal{J} can differ by at most m . Also in the case $n = 2$ we have

$$p_1 = (m + q_2 - p_2) + q_1 = (m + q_1 - p_2) + q_2 \geq \max\{q_1, q_2\}, \quad (15.10)$$

and similarly for p_2 .

Likewise we can redefine \mathcal{K} as the smallest subspace containing \mathcal{V} that is closed under the action of $\mathbf{\Gamma}_1, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \dots, \mathbf{\Lambda}_n$ and redefine

$$\mathcal{E} = \mathbf{\Gamma}_1\mathcal{K}, \quad \mathcal{J} = \mathbf{\Gamma}_2\mathcal{K}, \quad \mathcal{P}_j = \mathbf{\Lambda}_j\mathcal{K}, \quad j = 1, 2, \dots, n. \quad (15.11)$$

Let v be the dimension of \mathcal{V} , p_j be the dimension of \mathcal{P}_j , $j = 1, 2, \dots, n$, and let q_1 and q_2 represent the dimensions of \mathcal{E} and \mathcal{J} . The total dimension of the vector space \mathcal{K} is therefore

$$h = q_1 + q_2 = v + p_1 + p_2 + \dots + p_n. \quad (15.12)$$

The space

$$\mathcal{V} \oplus [\mathbf{\Lambda}_1(\mathcal{E})] \oplus [\mathbf{\Lambda}_2(\mathcal{E})] \oplus \dots \oplus [\mathbf{\Lambda}_n(\mathcal{E})] \quad (15.13)$$

certainly contains \mathcal{V} , and is closed under $\mathbf{\Gamma}_1$ (because it contains \mathcal{E}) and is closed under $\mathbf{\Lambda}_j$ for each j . It therefore must be \mathcal{K} and $\mathbf{\Lambda}_j(\mathcal{E})$ which has at most dimension q_1 must be \mathcal{P}_j . Thus for each j we have the inequality

$$p_j \leq q_1, \quad (15.14)$$

and summing these over j we obtain

$$q_2 \leq v + (n-1)q_1. \quad (15.15)$$

Similarly since

$$\mathcal{K} = \mathcal{V} \oplus [\mathbf{\Lambda}_1(\mathcal{J})] \oplus [\mathbf{\Lambda}_2(\mathcal{J})] \oplus \dots \oplus [\mathbf{\Lambda}_n(\mathcal{J})], \quad (15.16)$$

we obtain the inequalities

$$p_j \leq q_2, \quad q_1 \leq v + (n-1)q_2. \quad (15.17)$$

When $n = 2$ the constraints (15.15) and (15.17) imply that the dimensions of the subspaces \mathcal{E} and \mathcal{J} can differ by at most v . Also in the case $n = 2$ we have

$$p_1 = (q_2 - p_2) + q_1 - v = (q_1 - p_2) + q_2 - v \geq \max\{q_1, q_2\} - v, \quad (15.18)$$

with a similar inequality for p_2 .

16 Expressions for the numerator and denominator in the rational function

Assume that a $Z(n)$ subspace collection, with $m = 1$ has been pruned. Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{q_1+1}$ be a basis for $\mathcal{U} \oplus \mathcal{E}$ with \mathbf{w}_1 in \mathcal{U} and $\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_{q_1+1}$ in \mathcal{E} . In this basis $(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_1)\mathbf{\Lambda}_i(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_1)$ is represented by a $(q_1 + 1) \times (q_1 + 1)$ matrix \mathbf{A}_i , and since the $\mathbf{\Lambda}_i$ sum up to the identity operator it follows that

$$\sum_{i=1}^n \mathbf{A}_i = \mathbf{I}. \quad (16.1)$$

Also, because the subspace is pruned, $\mathbf{\Lambda}_i(\mathcal{U} \oplus \mathcal{E})$ can be identified with \mathcal{P}_i which implies the matrix \mathbf{A}_i must have at most rank p_i . It is exactly p_i if $\mathcal{P}_i \cap \mathcal{J} = 0$. The formula (9.1) for the Z -function implies

$$1/Z(z_1, z_2, \dots, z_n) = \mathbf{e}_1 \cdot \left[\sum_{i=1}^n z_i \mathbf{A}_i \right]^{-1} \mathbf{e}_1, \quad (16.2)$$

where \mathbf{e}_1 is the $q_1 + 1$ component unit vector $[1, 0, 0, \dots, 0]^T$. Hence, following the argument given in Section 29.2 of Milton (2002), $Z(z_1, z_2, \dots, z_n)$ can be expressed in the form (10.1) with numerator

$$p(z_1, z_2, \dots, z_n) = \det \left[\sum_{i=1}^n z_i \mathbf{A}_i \right] = \sum_{a_1, a_2, \dots, a_n} \alpha_{a_1 a_2 \dots a_n} z_1^{a_1} z_2^{a_2} \dots z_n^{a_n}, \quad (16.3)$$

of degree $1 + q_1$, in which the sum extends over all a_1, a_2, \dots, a_n with

$$\sum_{i=1}^n a_i = 1 + q_1, \quad 0 \leq a_i \leq p_i \quad \text{for } i = 1, 2, \dots, n. \quad (16.4)$$

Typically one expects that the maximum power of z_i in this polynomial will be the rank of \mathbf{A}_i . However, for example, note that for the matrices

$$\mathbf{M}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{M}_2 = \mathbf{I} - \mathbf{M}_1, \quad (16.5)$$

the maximum power of z_1 in

$$\det[z_1 \mathbf{M}_1 + z_2 \mathbf{M}_2] = \det[(z_1 - z_2) \mathbf{M}_1 + z_2 \mathbf{I}] = z_2[z_2^2 + 2z_2(z_1 - z_2)] \quad (16.6)$$

is 1 while \mathbf{M}_1 has rank 2.

Next let $\mathbf{w}_1, \mathbf{w}_{q_1+2}, \dots, \mathbf{w}_h$ be a basis for $\mathcal{U} \oplus \mathcal{J}$ with \mathbf{w}_1 in \mathcal{U} and $\mathbf{w}_{q_1+2}, \mathbf{w}_{q_1+3}, \dots, \mathbf{w}_h$ in \mathcal{J} . In this basis $(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{\Lambda}_i(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)$ is represented by a $(q_2 + 1) \times (q_2 + 1)$ matrix \mathbf{B}_i , and since the $\mathbf{\Lambda}_i$ sum up to the identity operator it follows that

$$\sum_{i=1}^n \mathbf{B}_i = \mathbf{I}. \quad (16.7)$$

Also, because the subspace is pruned, $\mathbf{\Lambda}_i(\mathcal{U} \oplus \mathcal{J})$ can be identified with \mathcal{P}_i which implies the matrix \mathbf{B}_i must have rank at most p_i . It is exactly p_i if $\mathcal{P}_i \cap \mathcal{E} = 0$. The formula (4.3) for the Z -function implies

$$Z(z_1, z_2, \dots, z_n) = \mathbf{e}_2 \cdot \left[\sum_{i=1}^n \mathbf{B}_i / z_i \right]^{-1} \mathbf{e}_2, \quad (16.8)$$

where \mathbf{e}_2 is the $q_2 + 1$ component unit vector $[1, 0, 0, \dots, 0]^T$. The denominator of this expression, as a polynomial in the variables $1/z_i$, is

$$\det \left[\sum_{i=1}^n \mathbf{B}_i / z_i \right] = \sum_{b_1, b_2, \dots, b_n} \beta_{b_1 b_2 \dots b_n} / z_1^{b_1} z_2^{b_2} \dots z_n^{b_n}, \quad (16.9)$$

in which the sum extends over all b_1, b_2, \dots, b_n with

$$\sum_{i=1}^n b_i = 1 + q_2, \quad 0 \leq b_i \leq p_i \quad \text{for } i = 1, 2, \dots, n. \quad (16.10)$$

Consequently, for the denominator in the expression (10.1) for $Z(z_1, z_2, \dots, z_n)$, we can make the identification

$$q(z_1, z_2, \dots, z_n) = \sum_{b_1, b_2, \dots, b_n} \beta_{b_1 b_2 \dots b_n} z_1^{p_1 - b_1} z_2^{p_2 - b_2} \dots z_n^{p_n - b_n}, \quad (16.11)$$

which is a polynomial of degree $h - (1 + q_2) = q_1$. Furthermore the identities (16.1) and (16.7) imply the polynomial p and q satisfy the normalization (10.3), i.e.

$$\sum_{a_1, a_2, \dots, a_n} \alpha_{a_1 a_2 \dots a_n} = 1, \quad \sum_{b_1, b_2, \dots, b_n} \beta_{b_1 b_2 \dots b_n} = 1. \quad (16.12)$$

17 The correspondence between rational functions of one variable and $Z(2)$ subspace collections with $m = 1$

In the case $m = 1$ and $n = 2$ there are two cases to consider. When the dimension of h is even, $h = 2d$, then in order to satisfy the inequalities (15.6), (15.7) and (15.9) the subspaces \mathcal{E} and \mathcal{J} must have dimension d and $d - 1$ or vice versa and the subspaces \mathcal{P}_1 and \mathcal{P}_2 must have dimension d . Without loss of generality, by making a duality transformation if necessary, let us suppose \mathcal{E} has dimension $d - 1$. Given $\mathbf{u} \in \mathcal{U}$ let us take as our basis for \mathcal{H} the vectors

$$\mathbf{v}_{2j-1} = (\mathbf{\Gamma}_1 \mathbf{\Lambda}_1)^{j-1} \mathbf{u}, \quad \mathbf{v}_{2j} = (\mathbf{\Lambda}_1 \mathbf{\Gamma}_1)^{j-1} \mathbf{\Lambda}_1 \mathbf{u}, \quad j = 1, 2, \dots, d, \quad (17.1)$$

so that

$$\mathbf{v}_1 = \mathbf{u}, \quad \mathbf{v}_{2j} = \mathbf{\Lambda}_1 \boldsymbol{\nu}_{2j-1}, \quad j = 1, 2, \dots, d, \quad \mathbf{v}_{2j+1} = \mathbf{\Gamma}_1 \boldsymbol{\nu}_{2j-1}, \quad j = 1, 2, \dots, d-1. \quad (17.2)$$

These fields are independent since if they were not we could prune the subspace collection. The vectors $\mathbf{v}_{2j+1}, j = 1, 2, \dots, d-1$, which number $d-1$, must form a basis for \mathcal{E} and so it follows that

$$\mathbf{\Gamma}_1 \mathbf{v}_{2d} = \sum_{i=1}^{d-1} \gamma_i \mathbf{v}_{2i+1}. \quad (17.3)$$

Also we have

$$\mathbf{\Gamma}_0 \mathbf{v}_1 = \mathbf{v}_1, \quad \mathbf{\Gamma}_0 \mathbf{v}_{2j} = \delta_j \mathbf{v}_1, \quad j = 1, 2, \dots, d, \quad \mathbf{\Gamma}_0 \mathbf{v}_{2j+1} = 0, \quad j = 1, 2, \dots, d-1. \quad (17.4)$$

The $2d-1$ constants $\gamma_1, \dots, \gamma_{d-1}$ and $\delta_1, \dots, \delta_d$ characterize the geometry of the subspace collection. The field $\mathbf{e} + \mathbf{E}$ must have the expansion

$$\mathbf{e} + \mathbf{E} = \sum_{i=1}^d a_i \mathbf{v}_{2i-1}, \quad (17.5)$$

and consequently, setting $z_2 = 1$

$$\mathbf{j} + \mathbf{J} = [\mathbf{I} + (z_1 - 1)\mathbf{\Lambda}_1](\mathbf{e} + \mathbf{E}) = \sum_{i=1}^d a_i \mathbf{v}_{2i-1} + (z_1 - 1) \sum_{i=1}^d a_i \mathbf{v}_{2i}. \quad (17.6)$$

Since $\mathbf{\Gamma}_1(\mathbf{j} + \mathbf{J}) = 0$ we arrive at the equations

$$\begin{aligned} 0 &= \sum_{i=2}^d a_i \mathbf{v}_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i \mathbf{v}_{2i+1} + (z_1 - 1) \sum_{i=1}^{d-1} a_d \gamma_i \mathbf{v}_{2i+1} \\ &= \sum_{i=1}^{d-1} [a_{i+1} + a_i(z_1 - 1) + \gamma_i a_d(z_1 - 1)] \mathbf{v}_{2i+1}. \end{aligned} \quad (17.7)$$

implying

$$a_{i+1} + a_i(z_1 - 1) + \gamma_i a_d(z_1 - 1) = 0, \quad i = 1, \dots, d-1. \quad (17.8)$$

Choosing a normalization with $a_d = (1 - z_1)^{d-1}$ these equations are solved with

$$a_i = (1 - z_1)^{i-1} - \sum_{j=i}^{d-1} \gamma_{d-1+i-j} (1 - z_1)^j. \quad (17.9)$$

Since

$$\mathbf{\Gamma}_0(\mathbf{e} + \mathbf{E}) = a_1 \mathbf{v}_1, \quad \mathbf{\Gamma}_0(\mathbf{j} + \mathbf{J}) = [a_1 + (z_1 - 1) \sum_{i=1}^d \delta_i a_i] \mathbf{v}_1, \quad (17.10)$$

we obtain

$$Z(z_1, 1) = 1 + \frac{(z_1 - 1) \sum_{i=1}^d \delta_i a_i}{a_1}. \quad (17.11)$$

Conversely suppose we are given a rational function $Z(z_1, 1)$ with a denominator of degree at most $d-1$ and a numerator of degree at most d satisfying $Z(1, 1) = 1$. It can be expressed in the form

$$Z(z_1, 1) = \frac{p(z_1, 1)}{q(z_1, 1)} = 1 - \frac{\sum_{j=0}^{d-1} t_j (1 - z_1)^{j+1}}{1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j}. \quad (17.12)$$

Comparing this with (17.11) we can make the identifications

$$\begin{aligned}
1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j &= a_1 = 1 - \sum_{j=1}^{d-1} \gamma_{d-j} (1 - z_1)^j, \\
- \sum_{j=0}^{d-1} t_j (1 - z_1)^{j+1} &= (z_1 - 1) \sum_{i=1}^d \delta_i a_i \\
&= - \sum_{j=0}^{d-1} \delta_{j+1} (1 - z_1)^{j+1} + \sum_{j=0}^{d-1} \sum_{i=1}^j \delta_i \gamma_{d-1+i-j} (1 - z_1)^{j+1}, \quad (17.13)
\end{aligned}$$

which imply

$$s_j = \gamma_{d-j}, \quad t_0 = \delta_1, \quad t_j = \delta_{j+1} - \sum_{i=1}^j \delta_i \gamma_{d-1+i-j} \quad j = 1, \dots, d-1. \quad (17.14)$$

Given the coefficients s and t we can inductively uniquely determine the coefficients γ and δ :

$$\gamma_j = s_{d-j}, \quad \delta_1 = t_0, \quad \delta_{j+1} = t_j + \sum_{i=1}^j \delta_i s_{1+j-i} \quad j = 1, \dots, d-1. \quad (17.15)$$

On the other hand when the dimension of h is odd, $h = 2d-1$, then in order to satisfy the inequalities (15.6), (15.7) and (15.9) the subspaces \mathcal{E} and \mathcal{J} must have dimension $d-1$ and the subspaces \mathcal{P}_1 and \mathcal{P}_2 must have dimension $d-1$ and d or vice versa. Without loss of generality let us suppose \mathcal{P}_1 has dimension $d-1$. Given $\mathbf{u} \in \mathcal{U}$ let us take as our basis for \mathcal{H} the vectors

$$\mathbf{v}_{2j-1} = (\mathbf{\Gamma}_1 \mathbf{\Lambda}_1)^{j-1} \mathbf{u}, \quad j = 1, 2, \dots, d-1, \quad \mathbf{v}_{2j} = (\mathbf{\Lambda}_1 \mathbf{\Gamma}_1)^{j-1} \mathbf{\Lambda}_1 \mathbf{u}, \quad j = 1, 2, \dots, d, \quad (17.16)$$

which satisfy

$$\mathbf{v}_1 = \mathbf{u}, \quad \mathbf{v}_{2j} = \mathbf{\Lambda}_1 \mathbf{v}_{2j-1}, \quad \mathbf{v}_{2j+1} = \mathbf{\Gamma}_1 \mathbf{v}_{2j}, \quad j = 1, 2, \dots, d-1. \quad (17.17)$$

Again these fields are independent since if they were not we could prune the subspace collection. The vectors $\mathbf{v}_{2j}, j = 1, 2, \dots, d-1$, which number $d-1$, must form a basis for \mathcal{P}_1 and so it follows that

$$\mathbf{\Lambda}_1 \mathbf{v}_{2d-1} = \sum_{i=1}^{d-1} \gamma_i \mathbf{v}_{2i}. \quad (17.18)$$

Also we have

$$\mathbf{\Gamma}_0 \mathbf{v}_1 = \mathbf{v}_1, \quad \mathbf{\Gamma}_0 \mathbf{v}_{2j} = \delta_j \mathbf{v}_1, \quad j = 1, 2, \dots, d-1, \quad \mathbf{\Gamma}_0 \mathbf{v}_{2j+1} = 0, \quad j = 1, 2, \dots, d-1. \quad (17.19)$$

The $2d-2$ constants $\gamma_1, \dots, \gamma_{d-1}$ and $\delta_1, \dots, \delta_{d-1}$ characterize the geometry of the subspace collection. The field $\mathbf{e} + \mathbf{E}$ has the expansion (17.5) and so, with $z_2 = 1$,

$$\mathbf{j} + \mathbf{J} = [\mathbf{I} + (z_1 - 1) \mathbf{\Lambda}_1] (\mathbf{e} + \mathbf{E}) = \sum_{i=1}^d a_i \mathbf{v}_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i \mathbf{v}_{2i} + (z_1 - 1) \sum_{i=1}^{d-1} a_d \gamma_i \mathbf{v}_{2i}. \quad (17.20)$$

Since $\Gamma_1(\mathbf{j} + \mathbf{J}) = 0$ we arrive at the equations

$$\begin{aligned} 0 &= \sum_{i=2}^d a_i \mathbf{v}_{2i-1} + (z_1 - 1) \sum_{i=1}^{d-1} a_i \mathbf{v}_{2i+1} + (z_1 - 1) \sum_{i=1}^{d-1} a_d \gamma_i \mathbf{v}_{2i+1} \\ &= \sum_{i=1}^{d-1} [a_{i+1} + a_i(z_1 - 1) + \gamma_i a_d(z_1 - 1)] \mathbf{v}_{2i+1}, \end{aligned} \quad (17.21)$$

implying (17.8) which has the solution (17.9). Since

$$\Gamma_0(\mathbf{e} + \mathbf{E}) = a_1 \mathbf{v}_1, \quad \Gamma_0(\mathbf{j} + \mathbf{J}) = [a_1 + (z_1 - 1) \sum_{i=1}^{d-1} \delta_i (a_i + a_d \gamma_i)] \mathbf{v}_1 = [a_1 - \sum_{i=1}^{d-1} \delta_i a_{i+1}] \mathbf{v}_1, \quad (17.22)$$

we obtain

$$Z(z_1, 1) = 1 - \frac{\sum_{i=1}^{d-1} \delta_i a_{i+1}}{a_1}. \quad (17.23)$$

Conversely suppose we are given a rational function $Z(z_1, 1)$ with a denominator of degree at most $d - 1$ and a numerator of degree at most $d - 1$. It can be expressed in the form

$$Z(z_1, 1) = 1 - \frac{\sum_{j=1}^{d-1} t_j (1 - z_1)^j}{1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j}. \quad (17.24)$$

Comparing this with (17.23) we can make the identifications

$$\begin{aligned} 1 - \sum_{j=1}^{d-1} s_j (1 - z_1)^j &= a_1 = 1 - \sum_{j=1}^{d-1} \gamma_{d-j} (1 - z_1)^j, \\ \sum_{j=1}^{d-1} t_j (1 - z_1)^j &= \sum_{i=1}^{d-1} \delta_i a_{i+1} \\ &= \sum_{j=1}^{d-1} \delta_j (1 - z_1)^j - \sum_{j=2}^{d-1} \sum_{i=1}^{j-1} \delta_i \gamma_{d+i-j} (1 - z_1)^j, \end{aligned} \quad (17.25)$$

which imply

$$s_j = \gamma_{d-j}, \quad j = 1, \dots, d-1, \quad t_1 = \delta_1, \quad t_j = \delta_j - \sum_{i=1}^{j-1} \delta_i \gamma_{d+i-j} \quad j = 2, \dots, d-1. \quad (17.26)$$

Given the coefficients s and t we can inductively uniquely determine the coefficients γ and δ :

$$\gamma_j = s_{d-j}, \quad j = 1, \dots, d-1, \quad \delta_1 = t_1, \quad \delta_j = t_j + \sum_{i=1}^j \delta_i s_{j-i} \quad j = 2, \dots, d-1. \quad (17.27)$$

One can see from this analysis that there can be more than one pruned subspace collection associated with a rational function $Z(z_1, 1)$. It may happen that one pruned

$Z(n)$ subspace collection gives rise to polynomials $p(z_1, 1) = f(z_1, 1)r(z_1, 1)$ and $q(z_1, 1) = g(z_1, 1)r(z_1, 1)$ while another pruned $Z(n)$ subspace collection gives rise to polynomials $p'(z_1, 1) = f(z_1, 1)r'(z_1, 1)$ and $q'(z_1, 1) = t(z_1, 1)r'(z_1, 1)$, so that both give rise to the same function $Z(z_1, 1)$. However there is a one-to-one correspondence if the pruned subspace collection is such that the polynomials $p(z_1, z_2)$ and $q(z_1, z_2)$ have no factor in common, and this correspondence is given by the above algorithm

18 On the correspondence between certain rational functions of two variables and $Z(3)$ subspace collections with $m = 1$

In the case $m = 1$ and $n = 3$ can we uniquely recover a generic subspace collection (modulo the linear transformations (13.9)) from knowledge of the rational function $Z(z_1, z_2, 1)$? The answer is no, but let us first provide a counting argument which suggests that, at least in the generic case, we can recover the subspace collection up to a finite number of possibilities. The counting argument is similar to that given in Section 29.2 of Milton (2002) but here we do not assume that the subspaces are orthogonal.

How many independent coefficients $\alpha_{a_1 a_2 a_3}$ are there in a polynomial

$$p(z_1, z_2, 1) = \sum_{a_1, a_2, a_3} \alpha_{a_1 a_2 a_3} z_1^{a_1} z_2^{a_2}, \quad (18.1)$$

that satisfies

$$a_1 + a_2 + a_3 = 1 + q_1, \quad 0 \leq a_i \leq p_i \leq 1 + q_1, \quad i = 1, 2, 3? \quad (18.2)$$

Without loss of generality, following Section 29.2 of Milton (2002), let us suppose that $p_1 \geq p_2 \geq p_3$. With a_1 fixed in the regime $0 \leq a_1 < 1 + q_1 - p_2$, the constant a_2 can take integer values from $a_2 = 1 + q_1 - a_1 - p_3$ (where $a_3 = p_3$) to $a_2 = p_2$, that is, a total of $p_2 + p_3 + a_1 - q_1$ different values. With a_1 fixed in the regime $1 + q_1 - p_2 \leq a_1 < 1 + q_1 - p_3$, the constant a_2 can take integer values from $a_2 = 1 + q_1 - a_1 - p_3$ (where $a_3 = p_3$) to $a_2 = 1 + q_1 - a_1$ (where $a_3 = 0$) that is, a total of $p_3 + 1$ different values. Finally, with a_1 fixed in the regime $1 + q_1 - p_3 \leq a_1 \leq p_1$, the constant a_2 can take integer values from $a_2 = 0$ to $a_2 = 1 + q_1 - a_1$ (where $a_3 = 0$), that is, a total of $2 + q_1 - a_1$ different values. Therefore the total number of coefficients in the polynomial is

$$\begin{aligned} & \sum_{a_1=0}^{q_1-p_2} (p_2 + p_3 + a_1 - q_1) + \sum_{a_1=1+q_1-p_2}^{q_1-p_3} (p_3 + 1) + \sum_{a_1=1+q_1-p_3}^{p_1} (2 + q_1 - a_1) \\ &= (q_1 - p_2 + 1)(p_2 + p_3 - q_1) + \frac{1}{2}(q_1 - p_2 + 1)(q_1 - p_2) + (p_2 - p_3)(p_3 + 1) \\ & \quad + (p_1 + p_3 - q_1)(2 + p_3) - \frac{1}{2}((p_1 + p_3 - q_1)(p_1 + p_3 - q_1 + 1)) \\ &= k_1 + 1, \end{aligned} \quad (18.3)$$

where

$$k_1 = [2(1 + q_1)q_2 - p_1^2 - p_2^2 - p_3^2 + h]/2, \quad (18.4)$$

in which $h = p_1 + p_1 + p_3$ and $q_2 = h - 1 - q_1$. These coefficients are not all independent since, from (16.12) the $\alpha_{a_1 a_2 a_3}$ must sum to one. Subtracting this constraint gives k_1 independent coefficients.

Similarly in a polynomial

$$q(z_1, z_2, 1) = \sum_{b_1, b_2, b_3} \beta_{b_1 b_2 b_3} z_1^{p_1 - b_1} z_2^{p_2 - b_2}, \quad (18.5)$$

that satisfies

$$b_1 + b_2 + b_3 = 1 + q_2, \quad 0 \leq a_i \leq p_i \leq 1 + q_2, \quad i = 1, 2, 3, \quad \sum_{b_1, b_2, b_3} \beta_{b_1 b_2 b_3} = 1, \quad (18.6)$$

there are a total of

$$k_2 = [2(1 + q_2)q_1 - p_1^2 - p_2^2 - p_3^2 + h]/2 \quad (18.7)$$

independent coefficients. Hence the total number of independent coefficients in the rational function

$$Z(z_1, z_2, 1) = \frac{p(z_1, z_2, 1)}{q(z_1, z_2, 1)} \quad (18.8)$$

is

$$k_1 + k_2 = (1 + q_1)q_2 + (1 + q_2)q_1 - p_1^2 - p_2^2 - p_3^2 + h = h^2 - p_1^2 - p_2^2 - p_3^2 - q_1^2 - q_2^2. \quad (18.9)$$

Now how many parameters describe a $Z(n)$ subspace collection, when the spaces \mathcal{U} , \mathcal{E} , \mathcal{J} , \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 have dimensions 1, q_1 , q_2 , p_1 , p_2 , and p_3 , with $1 + q_1 + q_2 = p_1 + p_2 + p_3 = h$? Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_h$ be a basis for \mathcal{H} with \mathbf{w}_1 in \mathcal{U} , $\mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_{q_1+1}$ in \mathcal{E} , and $\mathbf{w}_{q_1+2}, \mathbf{w}_{q_1+3}, \dots, \mathbf{w}_h$ in \mathcal{J} . Recall that it requires $s(d - s)$ parameters to describe the orientation of a subspace of dimension s in a space of dimension d . Therefore, it requires

$$p_1(h - p_1) + (h - p_2)p_2 + (h - p_3)p_3 = h^2 - p_1^2 - p_2^2 - p_3^2 \quad (18.10)$$

parameters to describe the orientation of the subspaces \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 with respect to this basis. However some of these subspace collections are equivalent, linked through transformations of the form (13.9). If respect to this basis \mathbf{C} is represented by a matrix with block form

$$\mathbf{C} = \begin{pmatrix} c & 0 & 0 \\ 0 & \mathbf{C}_1 & 0 \\ 0 & 0 & \mathbf{C}_2 \end{pmatrix}, \quad (18.11)$$

where c is a scalar, while \mathbf{C}_1 and \mathbf{C}_2 are $q_1 \times q_1$ and $q_2 \times q_2$ matrices, then it will leave the subspaces \mathcal{U} , \mathcal{E} and \mathcal{J} unchanged. The transformation $\mathbf{C} = a\mathbf{I}$ leaves all subspaces unchanged for any scalar $a \neq 0$, and so to factor out such trivial transformations we should choose $c = 1$. The number of remaining independent parameters in \mathbf{C} is then $q_1^2 + q_2^2$. Subtracting these from (18.10) we see that the number of parameters describing the $Z(n)$ subspace collection is

$$h^2 - p_1^2 - p_2^2 - p_3^2 - q_1^2 - q_2^2 = k_1 + k_2. \quad (18.12)$$

The precise agreement between the number of coefficients in the rational function and the number of parameters describing the $Z(n)$ subspace collection is curious (since it

holds for all q_1, q_2, p_1, p_2 , and p_3 , with $1 + q_1 + q_2 = p_1 + p_2 + p_3 = h$). Despite this coincidence we now show that it is not possible to uniquely recover a generic subspace collection (modulo the linear transformations (13.9)) from knowledge of the associated rational function $Z(z_1, z_2, 1)$.

Let us consider a subspace collection with $h = 5, q_1 = q_2 = 2, p_1 = p_2 = 1, p_3 = 3$ giving $k_1 + k_2 = 6$ according to the formula (18.9). Given $\mathbf{u} \in \mathcal{U}$ we choose as our basis the vectors

$$\mathbf{v}_0 = \mathbf{u}, \quad \mathbf{v}_1 = \Lambda_1 \mathbf{u}, \quad \mathbf{v}_2 = \Lambda_2 \mathbf{u}, \quad \mathbf{v}_3 = \Gamma_1 \Lambda_1 \mathbf{u}, \quad \mathbf{v}_4 = \Gamma_1 \Lambda_2 \mathbf{u}, \quad (18.13)$$

with the closure relations

$$\begin{aligned} \Lambda_1 \mathbf{v}_3 &= \gamma_1 \mathbf{v}_1, & \Lambda_2 \mathbf{v}_3 &= \gamma_2 \mathbf{v}_2, & \Lambda_1 \mathbf{v}_4 &= \gamma_3 \mathbf{v}_1, & \Lambda_2 \mathbf{v}_4 &= \gamma_4 \mathbf{v}_1, \\ \Gamma_0 \mathbf{v}_1 &= \delta_1 \mathbf{v}_0, & \Gamma_0 \mathbf{v}_2 &= \delta_2 \mathbf{v}_0, \end{aligned} \quad (18.14)$$

expressed in terms of the 6 parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1$, and δ_2 which describe the subspace collection. The question is: can one uniquely recover these six parameters from $Z(z_1, z_2, 1)$? Although the following analysis extends easily to the case of arbitrary γ_1 and γ_4 let us assume, for simplicity, that $\gamma_1 = \gamma_4 = 0$ and ask whether one can recover the remaining four parameters. The field $\mathbf{e} + \mathbf{E}$ must have the expansion

$$\mathbf{e} + \mathbf{E} = a_0 \mathbf{v}_0 + a_1 \mathbf{v}_3 + a_2 \mathbf{v}_4, \quad (18.15)$$

and consequently, setting $z_3 = 1$,

$$\begin{aligned} \mathbf{j} + \mathbf{J} &= [\mathbf{I} + (z_1 - 1)\Lambda_1 + (z_2 - 1)\Lambda_2](\mathbf{e} + \mathbf{E}) \\ &= a_0 \mathbf{v}_0 + a_1 \mathbf{v}_3 + a_2 \mathbf{v}_4 + (z_1 - 1)(a_0 + a_2 \gamma_3) \mathbf{v}_1 + (z_2 - 1)(a_0 + a_1 \gamma_2) \mathbf{v}_2 \end{aligned} \quad (18.16)$$

Since $\Gamma_1(\mathbf{j} + \mathbf{J}) = 0$ we arrive at the equations

$$0 = a_1 \mathbf{v}_3 + a_2 \mathbf{v}_4 + (z_1 - 1)(a_0 + a_2 \gamma_3) \mathbf{v}_3 + (z_2 - 1)(a_0 + a_1 \gamma_2) \mathbf{v}_4, \quad (18.17)$$

implying

$$a_1 + (z_1 - 1)(a_0 + a_2 \gamma_3) = 0, \quad a_2 + (z_2 - 1)(a_0 + a_1 \gamma_2) = 0. \quad (18.18)$$

These equations have as a solution,

$$\begin{aligned} a_0 &= 1 - (z_1 - 1)(z_2 - 1)\gamma_2\gamma_3, \\ a_1 &= \gamma_3(z_1 - 1)(z_2 - 1) - (z_1 - 1), \\ a_2 &= \gamma_2(z_1 - 1)(z_2 - 1) - (z_2 - 1). \end{aligned} \quad (18.19)$$

Since

$$\Gamma_0(\mathbf{e} + \mathbf{E}) = a_0 \mathbf{v}_0, \quad \Gamma_0(\mathbf{j} + \mathbf{J}) = [a_0 + (z_1 - 1)(a_0 + a_2 \gamma_3)\delta_1 + (z_2 - 1)(a_0 + a_1 \gamma_2)\delta_2] \mathbf{v}_0, \quad (18.20)$$

we obtain

$$\begin{aligned} Z(z_1, z_2, 1) &= 1 + \frac{(z_1 - 1)(a_0 + a_2 \gamma_3)\delta_1 + (z_2 - 1)(a_0 + a_1 \gamma_2)\delta_2}{a_0} \\ &= 1 + \frac{\delta_1(z_1 - 1) - \gamma_3\delta_1(z_1 - 1)(z_2 - 1) + \delta_2(z_2 - 1) - \gamma_2\delta_2(z_1 - 1)(z_2 - 1)}{1 - (z_1 - 1)(z_2 - 1)\gamma_2\gamma_3}. \end{aligned} \quad (18.21)$$

Given this function we can uniquely determine δ_1 and δ_2 from the coefficients of $(z_1 - 1)$ and $(z_2 - 1)$ in the numerator. Also from the coefficients of $(z_1 - 1)(z_2 - 1)$ in the numerator and denominator we can uniquely determine

$$t_1 = \gamma_2 \gamma_3, \quad t_2 = \gamma_3 \delta_1 + \gamma_2 \delta_2, \quad (18.22)$$

in terms of which there are two possible values of γ_2 , namely

$$\gamma_2 = \frac{t_3 \pm \sqrt{t_3^2 - 3t_1 \delta_1 \delta_2}}{2\delta_1}. \quad (18.23)$$

Thus we cannot uniquely recover the subspace collection parameters from $Z(z_1, z_2, 1)$.

It remains an open question, raised at the end of Section 29.2 of Milton (2002), as to whether in general one can uniquely recover the subspace collection parameters when, with respect to some inner product, the subspaces \mathcal{U} , \mathcal{E} and \mathcal{J} are mutually orthogonal, and the subspaces \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 are mutually orthogonal. These orthogonality constraints overdetermine the system of equations needed to recover the subspace collection parameters which provides some hope that we can recover them. It would be useful if one could uniquely recover the subspace collection parameters (the weight and normalization matrices introduced in Milton, 1987a, 1987b) from say the effective conductivity $\sigma_*(\sigma_1, \sigma_2, \sigma_3)$ of an isotropic composite of three isotropic phases having conductivities σ_1 , σ_2 , and σ_3 as then one could obtain the effective response tensor for coupled field problems. We will see in Chapter 9 of this book (Milton 2016) that the effective response tensor just depends on the weight and normalization matrices for the uncoupled conductivity problem.

19 Visualizing the poles and zeros of functions associated with orthogonal $Z(3)$ subspace collections when $m = 1$

For scalar functions $Z(z_1, z_2, z_3)$, associated with orthogonal $Z(3)$ subspace collections, satisfying the homogeneity, Herglotz, and normalization properties, the trajectories of their poles and zeros in (z_1, z_2, z_3) space, with z_1 , z_2 , and z_3 taking real values, have a beautiful visualization as trajectories on three interlinked hexagons: To obtain this visualization we follow Appendix C in Nicorovici, McPhedran, and Milton (1993): see also figure 5 in that paper.

First note that if we set $z_3 = 1$, then the poles and zeros of $Z(z_1, z_2, 1)$ lie in one of the three quadrants:

- The quadrant $z_1 \leq 0, z_2 \geq 0$;
- The quadrant $z_2 \leq 0, z_1 \geq 0$;
- The quadrant $z_1 \leq 0, z_2 \leq 0$.

Of course we can visualize the pole and zero trajectories by plotting them in this plane, but this has the disadvantage that the three variables z_1 , z_2 and z_3 are not treated in

a symmetric way, and the disadvantage that its hard to see what is happening when z_1 and/or z_2 is large, and it is hard to see what is happening near the origin $z_1 = z_2 = 0$ since the trajectories can bunch up there. To get around this we map each of the three quadrants to a hexagon. Given a quadrant, the point $z_1 = z_2 = 0$ gets blown up to form one edge of the hexagon; the two edges of the quadrant where z_1 or z_2 is zero, but not the other, get mapped to two other edges of the hexagon; the two “boundaries” of the quadrant where $|z_1|$ or $|z_2|$ is infinite but other is finite get mapped to two more edges of the hexagon; finally $z_1 = z_2 = \infty$ gets mapped to the final sixth edge of the hexagon. We remark that just as a pole trajectory can cross from one quadrant to another, so too can it jump from the boundary of one hexagon to the corresponding point on the boundary of another hexagon.

To be more precise, we introduce the three variables

$$t_1 = \frac{1}{1 + |z_2/z_3|}, \quad t_2 = \frac{1}{1 + |z_3/z_1|}, \quad t_3 = \frac{1}{1 + |z_1/z_2|}. \quad (19.1)$$

Clearly (t_1, t_2, t_3) takes values in the unit cube. It is confined to a surface within the unit cube as the three ratios $|z_2|/|z_3|$, $|z_3|/|z_1|$ and $|z_1|/|z_2|$ are not independent, but have product 1. The next step is to map these three variables onto three variables s_1 , s_2 and s_3 lying in the plane $s_1 + s_2 + s_3 = 0$ using the projection

$$s_1 = 2t_1 - t_2 - t_3, \quad s_2 = 2t_2 - t_3 - t_1, \quad s_3 = 2t_3 - t_1 - t_2. \quad (19.2)$$

Finally, we map these down to the x - y plane:

$$x = s_1, \quad y = (s_1 + 2s_2)/\sqrt{3}. \quad (19.3)$$

Some normalization is needed, so in the hexagon where z_1 is negative and z_2 and z_3 are positive, we plot $Z(z_1, z_2, z_3)/\sqrt{z_2 z_3}$; in the hexagon where z_2 is negative and z_1 and z_3 are positive, we plot $Z(z_1, z_2, z_3)/\sqrt{z_1 z_3}$; and in the hexagon where z_3 is negative and z_1 and z_2 are positive, we plot $Z(z_1, z_2, z_3)/\sqrt{z_1 z_2}$.

Figure 8 uses this approach to visualize the pole trajectory of a function $Z(z_1, z_2, z_3)$ associated with a $Z(3)$ -subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3, \quad (19.4)$$

where in this example \mathcal{H} is 12-dimensional; \mathcal{U} is one-dimensional; \mathcal{P}_1 is 3-dimensional; \mathcal{P}_2 is 6-dimensional; \mathcal{P}_3 is 3-dimensional. Note that as the subspace collection does not need pruning, the dimensions of \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 can be immediately read off from the figure by simply counting the number of pole paths on each hexagon: figures (a), (b), and (c) have 3, 6 and 3 pole paths corresponding to the dimensions of \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 , respectively. To understand this, first recognize that when z_2 and z_3 are fixed, and real and positive, $Z(z_1, z_2, z_3)$ is a Herglotz function of z_1 taking real positive values when $z_1 > 0$. Thus all its poles must be simple and located on the negative real z_1 -axis, i.e. on the hexagon (a). Also because the subspace is pruned $\Lambda_1(\mathcal{U} \oplus \mathcal{J})$ can be identified with \mathcal{P}_1 (Section 16), and hence the matrix \mathbf{C}_1 representing $\Lambda_1(\Gamma_0 + \Gamma_2)$ has rank p_1 . Then as \mathbf{C}_1 and $\mathbf{C}_1^T \mathbf{C}_1$ have equal rank (this well-known fact can easily be seen by showing that they have the same null-space), and as the subspace collection is orthogonal, it follows that the matrix

$\mathbf{B}_1 = \mathbf{C}_1^T \mathbf{C}_1$ representing $(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{\Lambda}_1(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)$ has exactly rank p_1 . Similarly the matrix \mathbf{A}_1 representing $(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)\mathbf{\Lambda}_1(\mathbf{\Gamma}_0 + \mathbf{\Gamma}_2)$ has exactly rank p_1 . Therefore the sum over a_1 in the numerator in (16.3), goes up to $a_1 = p_1$, while the sum in the denominator in (16.11), goes from 0 up to $b_1 = p_1$ or (when all the coefficients $\beta_{p_1 b_2 b_3}$ are zero) to $b_1 = p_1 - 1$: it cannot go only up to $b_1 = p_1 - 2$, since as a function of z_1 , $Z(z_1, z_2, z_3)/\sqrt{z_2 z_3}$ with fixed $z_2 > 0$ and fixed $z_3 > 0$ can only have a simple pole at $z_1 = \infty$. When the sum over b_1 goes up to $b_1 = p_1$, there are clearly p poles of the function $Z(z_1, z_2, z_3)/\sqrt{z_2 z_3}$ on the hexagon as z_1 varies with fixed $z_2 > 0$ and fixed $z_3 > 0$. When the sum over b_1 goes up to $b_1 = p_1 - 1$, there are still p poles of the function $Z(z_1, z_2, z_3)/\sqrt{z_2 z_3}$ on the hexagon as z_1 varies with fixed $z_2 > 0$ and fixed $z_3 > 0$ provided we count the pole at $z_1 = \infty$.

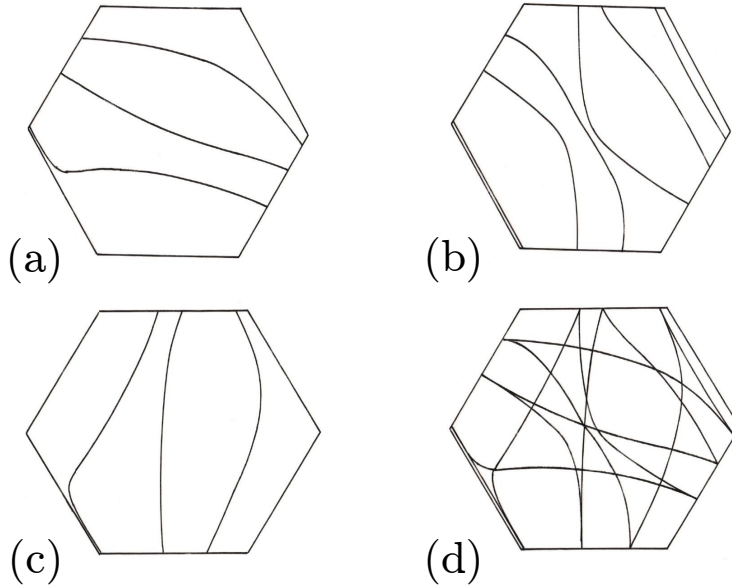


Figure 8: The pole trajectory of the function $Z(z_1, z_2, z_3)$ as visualized using the representation using three interlinked hexagons. The hexagon in (a) corresponds to real values of (z_1, z_2, z_3) where z_2 and z_3 have the same sign, but z_1 has the opposite sign. The hexagon in (b) corresponds to real values of (z_1, z_2, z_3) where z_1 and z_3 have the same sign, but z_2 has the opposite sign. The hexagon in (c) corresponds to real values of (z_1, z_2, z_3) where z_1 and z_2 have the same sign, but z_3 the opposite sign. By superimposing all three pictures one obtains (d) where the pole trajectory is like that of a billiard ball bouncing around a hexagonal table, following curved paths. The zero trajectory is similar, but for clarity we chose not to include it. Note that the dimensions 3, 6 and 3 of the subspaces \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{P}_3 can be immediately read off from the number of paths crossing the hexagons in (a), (b) and (c).

The dimension q_2 of the subspace \mathcal{J} can also generically be read off from the pole trajectories on the three hexagons. Consider the edge joining two of the hexagons that corresponds to the values $z_2 = 0$, and $z_3 = 1$ with $z_1 < 0$ varying. Then the only coefficients $\beta_{b_1 b_2 b_3}$ that can contribute to the denominator in (16.11) are those with $p_2 =$

b_2 . The first constraint in (16.10) then implies

$$b_1 + b_3 = 1 + q_2 - p_2. \quad (19.5)$$

So b_1 can only range from 0 up to the maximum of p_1 and $1 + q_2 - p_2 = p_1 + p_3 - q_1$. Note that according to the inequality (15.6), $q_1 \geq p_3 - 1$ so $1 + q_2 - p_2$ could be as large as $p_1 + 1$. If there are less than p_1 pole trajectories crossing this edge joining the hexagons, the number of these crossing pole trajectories should generically allow us to determine q_2 and hence q_1 , assuming p_1 , p_2 and p_3 have been determined from the number of pole trajectories on each hexagon. If there are exactly p_1 pole trajectories crossing the edge then q_2 could be p_3 or $p_3 + 1$. To determine which it is (or as an additional check on the value of q_2) we could look at pole trajectories, or zero trajectories, crossing other edges where the hexagons meet.

This visualization may be useful in finding other topological features of the trajectories, which hopefully could be connected with topological features of the subspace collections.

20 Normalization operations on subspace collections

Rational functions of a single variable may be expanded in continued fractions, which incorporate successively higher and higher order terms in the series expansion of the function about a point. The analogous procedure with subspace collections is achieved through normalization and reduction operations, subject to some technical assumptions. The associated functions are then linked, and provided the technical assumptions hold at each level, these links provide continued fractions for multivariate functions $\mathbf{Z}(z_1, z_2, \dots, z_n)$ and $\mathbf{Y}(z_1, z_2, \dots, z_n)$ incorporating matrices of increasingly high dimension at each level in the continued fraction.

The normalization and reduction operations are discussed in this and the next section. For more insight, in the case where the subspaces in the direct sums are orthogonal (see Milton 1987a, 1987b and Sections 19.2, 20.6 and 29.5 in Milton 2002).

Normalization reverses extension. Given a subspace collection

$$\mathcal{K} = \mathcal{E}' \oplus \mathcal{J}' = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, \quad (20.1)$$

define

$$\begin{aligned} \mathcal{H} &= \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, & \mathcal{E} &= \mathcal{E}' \cap \mathcal{H}, & \mathcal{J} &= \mathcal{J}' \cap \mathcal{H}, \\ \mathcal{U} &= \mathbf{\Pi}_2 \mathbf{\Gamma}'_1 \mathcal{V} = \mathbf{\Pi}_2 (\mathbf{I} - \mathbf{\Gamma}'_2) \mathcal{V} = \mathbf{\Pi}_2 \mathbf{\Gamma}'_2 \mathcal{V}, & \tilde{\mathcal{E}} &= \mathbf{\Gamma}'_1 \mathcal{V}, & \tilde{\mathcal{J}} &= \mathbf{\Gamma}'_2 \mathcal{V}, \end{aligned} \quad (20.2)$$

where $\mathbf{\Gamma}'_1$ and $\mathbf{\Gamma}'_2$ are the projections onto \mathcal{E}' and \mathcal{J}' , and $\mathbf{\Pi}_2$ is the projection onto \mathcal{H} .

We assume that the Y -problem has a unique solution when $\mathbf{L} = \mathbf{I}$ for $\mathbf{J}_1 \in \mathcal{V}$ given $\mathbf{E}_1 \in \mathcal{V}$. In other words, we assume that the equations

$$\begin{aligned} \mathbf{E}_1 + \mathbf{E}_2 &\in \mathcal{E}', & \mathbf{J}_1 + \mathbf{J}_2 &\in \mathcal{J}', & \mathbf{J}_2 &= \mathbf{E}_2, & \mathbf{E}_1, \mathbf{J}_1 &\in \mathcal{V}, & \mathbf{E}_2, \mathbf{J}_2 &\in \mathcal{H}, \\ \mathbf{E}_1 + \underline{\mathbf{E}}_2 &\in \mathcal{E}', & \underline{\mathbf{J}}_1 + \underline{\mathbf{J}}_2 &\in \mathcal{J}', & \underline{\mathbf{J}}_2 &= \underline{\mathbf{E}}_2, & \underline{\mathbf{J}}_1 &\in \mathcal{V}, & \underline{\mathbf{E}}_2, \underline{\mathbf{J}}_2 &\in \mathcal{H}, \end{aligned} \quad (20.3)$$

imply $\underline{\mathbf{J}}_1 = \mathbf{J}_1$. Subtracting these equations we see that

$$\mathbf{E} \equiv \mathbf{E}_2 - \underline{\mathbf{E}}_2 \in \mathcal{E}', \quad \mathbf{J} \equiv \mathbf{J}_1 + \mathbf{J}_2 - \underline{\mathbf{J}}_1 - \underline{\mathbf{J}}_2 \in \mathcal{J}', \quad \mathbf{J}_2 - \underline{\mathbf{J}}_2 = \mathbf{E}. \quad (20.4)$$

These imply

$$\mathbf{E} \in \mathcal{H}, \quad \mathbf{E} = \mathbf{J} - \mathbf{v}, \quad \text{where} \quad \mathbf{v} = \mathbf{J}_1 - \mathbf{J}_1. \quad (20.5)$$

The uniqueness assumption means that these equations imply $\mathbf{v} = 0$ (and if $\mathbf{v} = 0$ then necessarily $\mathbf{E} = \mathbf{J} = 0$ since \mathcal{E}' and \mathcal{J}' have no vector in common). The relation $\mathbf{E} = \mathbf{J} - \mathbf{v}$ with $\mathbf{E} \in \mathcal{E}' \cap \mathcal{H}$ implies

$$\mathbf{E} = -\mathbf{\Gamma}'_1 \mathbf{v}, \quad (20.6)$$

which will only have the trivial solution $\mathbf{v} = 0$ if and only if

$$\mathcal{H} \cap \tilde{\mathcal{E}} = 0 \quad \text{and} \quad \mathcal{V} \cap \mathcal{J}' = 0, \quad (20.7)$$

where the latter guarantees that $\mathbf{\Gamma}'_1 \mathbf{v} = 0$ implies $\mathbf{v} = 0$.

We also assume that the Y -problem has a unique solution when $\mathbf{L} = \mathbf{I}$ for $\mathbf{E}_1 \in \mathcal{V}$ given $\mathbf{J}_1 \in \mathcal{V}$. By similar analysis this is satisfied if and only if

$$\mathcal{H} \cap \tilde{\mathcal{J}} = 0 \quad \text{and} \quad \mathcal{V} \cap \mathcal{E}' = 0. \quad (20.8)$$

We now establish that

$$\mathcal{W} \equiv \tilde{\mathcal{E}} \oplus \tilde{\mathcal{J}} = \mathcal{V} \oplus \mathcal{U}. \quad (20.9)$$

First note that \mathcal{V} and \mathcal{U} have no vector in common since $\mathcal{U} \subset \mathcal{H}$, and similarly $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{J}}$ have no vector in common since $\mathcal{E}' \cap \mathcal{J}' = 0$. Clearly \mathcal{W} contains \mathcal{V} . To show it contains \mathcal{U} notice that

$$\mathcal{U} = \mathbf{\Pi}_2 \mathbf{\Gamma}'_1 \mathcal{V} = (\mathbf{I} - \mathbf{\Pi}_1) \mathbf{\Gamma}'_1 \mathcal{V} \subset \mathbf{\Gamma}'_1 \mathcal{V} \oplus \mathbf{\Pi}_1 \mathbf{\Gamma}'_1 \mathcal{V} \subset \tilde{\mathcal{E}} \oplus \mathcal{V} \subset \mathcal{W}. \quad (20.10)$$

Together these imply $\mathcal{V} \oplus \mathcal{U} \subset \mathcal{W}$. Finally we have

$$\tilde{\mathcal{E}} = \mathbf{\Gamma}'_1 \mathcal{V} = (\mathbf{\Pi}_1 + \mathbf{\Pi}_2) \mathbf{\Gamma}'_1 \mathcal{V} \subset \mathbf{\Pi}_1 \mathbf{\Gamma}'_1 \mathcal{V} \oplus \mathbf{\Pi}_2 \mathbf{\Gamma}'_1 \mathcal{V} \subset \mathcal{V} \oplus \mathcal{U}, \quad (20.11)$$

and similarly $\tilde{\mathcal{J}} \subset \mathcal{V} \oplus \mathcal{U}$. Together these imply $\mathcal{W} \subset \mathcal{V} \oplus \mathcal{U}$, establishing (20.9).

If \mathcal{V} has dimension m then $\tilde{\mathcal{E}}$ must also have dimension m since otherwise $\mathbf{\Gamma}'_1 \mathbf{v} = 0$ for some nonzero $\mathbf{v} \in \mathcal{V}$, implying $\mathbf{v} = \mathbf{\Gamma}'_2 \mathbf{v}$ which only has the solution $\mathbf{v} = 0$ since $\mathcal{V} \cap \mathcal{J}' = 0$. Similarly $\tilde{\mathcal{J}}$ must have dimension m and (20.9) then implies \mathcal{U} must have dimension m . The first condition in (20.7) implies

$$\mathcal{W} = \mathcal{U} \oplus \tilde{\mathcal{E}}, \quad (20.12)$$

since $\mathcal{U} \subset \mathcal{H}$ and $\tilde{\mathcal{E}}$ have no vector in common and are m -dimensional spaces contained in the $2m$ -dimensional space \mathcal{W} . Now any vector $\mathbf{E}' \in \mathcal{E}'$ has the unique decomposition

$$\mathbf{E}' = \mathbf{E}'_1 + \mathbf{P}, \quad \mathbf{E}'_1 \in \mathcal{V}, \quad \mathbf{P} \in \mathcal{H}, \quad (20.13)$$

and according to (20.12) \mathbf{E}'_1 has the unique decomposition

$$\mathbf{E}'_1 = -\mathbf{e} + \tilde{\mathbf{E}}, \quad \mathbf{e} \in \mathcal{U}, \quad \tilde{\mathbf{E}} \in \tilde{\mathcal{E}}. \quad (20.14)$$

So we have the decomposition

$$\mathbf{E}' = \tilde{\mathbf{E}} + \mathbf{E}, \quad (20.15)$$

where

$$\mathbf{E} = \mathbf{P} - \mathbf{e} = \mathbf{E}' - \tilde{\mathbf{E}} \in \mathcal{E}' \cap \mathcal{H} = \mathcal{E}. \quad (20.16)$$

Also the first condition in (20.7) implies $\tilde{\mathcal{E}}$ and $\mathcal{E} \subset \mathcal{H}$ have no vector in common, so the decomposition is unique. Therefore we conclude that

$$\mathcal{E}' = \tilde{\mathcal{E}} \oplus \mathcal{E}, \quad (20.17)$$

and similarly the first condition in (20.8) implies

$$\mathcal{J}' = \tilde{\mathcal{J}} \oplus \mathcal{J}. \quad (20.18)$$

These and (20.9) imply

$$\mathcal{K} = \mathcal{V} \oplus \mathcal{H} = \tilde{\mathcal{E}} \oplus \mathcal{E} \oplus \tilde{\mathcal{J}} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}, \quad (20.19)$$

and since \mathcal{U} , \mathcal{E} and \mathcal{J} are all contained in \mathcal{H} we conclude that

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n. \quad (20.20)$$

Now a given $\mathbf{E}'_1 \in \mathcal{V}$ has the unique decomposition (20.14). This defines the nonsingular operator $\mathbf{K} : \mathcal{V} \rightarrow \mathcal{U}$ such that $\mathbf{e} = \mathbf{K}\mathbf{E}'_1$. (It is nonsingular because \mathcal{V} and $\tilde{\mathcal{E}} \subset \mathcal{E}'$ have no nonzero vector in common.) Now given \mathbf{e} , consider the solution to

$$\mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{j} + \mathbf{J} = \mathbf{L}(\mathbf{e} + \mathbf{E}), \quad \text{where } \mathbf{L} = \sum_{i=1}^n z_i \mathbf{\Lambda}_i, \quad (20.21)$$

where $\mathbf{\Lambda}_i$ is the projection onto \mathcal{P}_i , and from the definition of \mathbf{Z} , $\mathbf{j} = \mathbf{Z}\mathbf{e}$. Since the second condition in (20.7) implies \mathcal{V} and $\tilde{\mathcal{J}}$ have no vector in common we have

$$\mathcal{W} = \mathcal{V} \oplus \tilde{\mathcal{J}}, \quad (20.22)$$

and consequently any $\mathbf{j} \in \mathcal{U}$ has the decomposition

$$\mathbf{j} = -\mathbf{J}'_1 + \tilde{\mathbf{J}}, \quad \mathbf{J}'_1 \in \mathcal{V}, \quad \tilde{\mathbf{J}} \in \tilde{\mathcal{J}}, \quad (20.23)$$

which defines the nonsingular operator $\mathbf{M} : \mathcal{U} \rightarrow \mathcal{V}$ such that $\mathbf{J}'_1 = \mathbf{M}\mathbf{j}$. Defining

$$\mathbf{E}'_2 = \mathbf{e} + \mathbf{E}, \quad \mathbf{J}'_2 = \mathbf{j} + \mathbf{J}, \quad (20.24)$$

we have

$$\begin{aligned} \mathbf{E}'_1 + \mathbf{E}'_2 &= \mathbf{E}'_1 + \mathbf{e} + \mathbf{E} = \tilde{\mathbf{E}} + \mathbf{E} \in \mathcal{E}', \\ \mathbf{J}'_1 + \mathbf{J}'_2 &= \mathbf{J}'_1 + \mathbf{j} + \mathbf{J} = \tilde{\mathbf{J}} + \mathbf{J} \in \mathcal{J}', \end{aligned} \quad (20.25)$$

and

$$\mathbf{J}'_1 = \mathbf{M}\mathbf{j} = \mathbf{M}\mathbf{Z}\mathbf{e} = \mathbf{M}\mathbf{Z}\mathbf{K}\mathbf{E}'_1, \quad (20.26)$$

which by definition of the associated Y -function implies

$$\mathbf{Y}(z_1, z_2, \dots, z_n) = \mathbf{M}\mathbf{Z}(z_1, z_2, \dots, z_n)\mathbf{K}. \quad (20.27)$$

This is analogous to the relation (20.29) in Milton (2002) obtained in the case where the subspaces are mutually orthogonal.

In particular by letting $z_1 = z_2 = \dots = z_n = 1$ we obtain

$$\mathbf{Y}(1, 1, \dots, 1) = \mathbf{MK}. \quad (20.28)$$

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are a basis for \mathcal{V} , and we choose $\mathbf{K}\mathbf{v}_1, \mathbf{K}\mathbf{v}_2, \dots, \mathbf{K}\mathbf{v}_m$ as our basis for \mathcal{U} then with these bases \mathbf{K} is represented by the identity matrix $\mathbf{K} = \mathbf{I}$ and (20.27) and (20.28) imply

$$\mathbf{Y}(z_1, z_2, \dots, z_n) = \mathbf{Y}(1, 1, \dots, 1)\mathbf{Z}(z_1, z_2, \dots, z_n). \quad (20.29)$$

21 Reduction operations on subspace collections

Extension is one way to go from a $Z(n)$ subspace collection to a $Y(n)$ subspace collection. Another way is through reduction, which has some features in common with normalization. Given a $Z(n)$ subspace collection

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_n, \quad (21.1)$$

let $\mathbf{\Gamma}_0$ be the projection onto \mathcal{U} , and let $\mathbf{\Lambda}_j$ be the projection onto \mathcal{P}_j . Define

$$\begin{aligned} \mathcal{K} &= \mathcal{E} \oplus \mathcal{J}, \quad \mathcal{P}'_j = \mathcal{P}_j \cap \mathcal{K} \quad \text{for } j = 1, 2, \dots, n, \\ \mathcal{V} &= (\mathbf{I} - \mathbf{\Gamma}_0)[\mathbf{\Lambda}_1\mathcal{U} \oplus \mathbf{\Lambda}_2\mathcal{U} \oplus \dots \oplus \mathbf{\Lambda}_n\mathcal{U}] \subset \mathcal{K}, \quad \tilde{\mathcal{P}}_j = \mathbf{\Lambda}_j\mathcal{U}. \end{aligned} \quad (21.2)$$

We now establish that

$$\mathcal{W} \equiv \tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 \oplus \dots \oplus \tilde{\mathcal{P}}_n = \mathcal{U} \oplus \mathcal{V}. \quad (21.3)$$

First note that \mathcal{V} and \mathcal{U} have no vector in common since $\mathcal{V} \subset \mathcal{K}$, and similarly the subspaces $\tilde{\mathcal{P}}_j$ have no vector in common since $\tilde{\mathcal{P}}_j \subset \mathcal{P}_j$. Clearly \mathcal{W} contains \mathcal{U} since the projections $\mathbf{\Lambda}_j$ sum to the identity. To show it contains \mathcal{V} note that

$$\mathcal{V} \subset \mathbf{\Lambda}_1\mathcal{U} \oplus \mathbf{\Lambda}_2\mathcal{U} \oplus \dots \oplus \mathbf{\Lambda}_n\mathcal{U} + \mathbf{\Gamma}_0[\mathbf{\Lambda}_1\mathcal{U} \oplus \mathbf{\Lambda}_2\mathcal{U} \oplus \dots \oplus \mathbf{\Lambda}_n\mathcal{U}] \subset \mathcal{W} + \mathcal{U} = \mathcal{W}. \quad (21.4)$$

Therefore we have that $\mathcal{U} \oplus \mathcal{V} \subset \mathcal{W}$. The converse inclusion that $\mathcal{W} \subset \mathcal{U} \oplus \mathcal{V}$ follows from the inclusion

$$\tilde{\mathcal{P}}_j = [\mathbf{\Gamma}_0 + (\mathbf{I} - \mathbf{\Gamma}_0)]\mathbf{\Lambda}_j\mathcal{U} \subset \mathcal{U} \oplus \mathcal{V}, \quad (21.5)$$

which establishes (21.3). Next, to establish that for all j ,

$$\mathcal{P}_j = \tilde{\mathcal{P}}_j \oplus \mathcal{P}'_j, \quad (21.6)$$

we need to assume that for all j

$$\tilde{\mathcal{P}}_j \cap \mathcal{K} = 0, \quad (21.7)$$

and that

$$\mathbf{\Lambda}_j\mathbf{u} = 0, \quad \mathbf{u} \in \mathcal{U} \quad (21.8)$$

only has the trivial solution $\mathbf{u} = 0$, i.e.

$$\mathcal{U} \cap (\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \dots \oplus \mathcal{P}_{j-1} \oplus \mathcal{P}_{j+1} \oplus \dots \oplus \mathcal{P}_n) = 0. \quad (21.9)$$

These conditions imply that

$$\mathcal{U} = \Gamma_0 \Lambda_j \mathcal{U}, \quad (21.10)$$

and hence that

$$\mathcal{U} \subset \Lambda_j \mathcal{U} \oplus (\mathbf{I} - \Gamma_0) \Lambda_j \mathcal{U}, \quad (21.11)$$

which in turn implies that

$$\mathcal{U} \subset \tilde{\mathcal{P}}_j + \mathcal{V}. \quad (21.12)$$

Then any vector $\mathbf{P} \in \mathcal{P}_j$ has the unique decomposition

$$\mathbf{P} = \mathbf{u} + \mathbf{K}, \quad \text{with } \mathbf{u} \in \mathcal{U}, \quad \mathbf{K} \in \mathcal{K}, \quad (21.13)$$

and according to (21.12), \mathbf{u} has the unique decomposition

$$\mathbf{u} = \mathbf{v} + \tilde{\mathbf{P}} \quad \text{with } \mathbf{v} \in \mathcal{V}, \quad \tilde{\mathbf{P}} \in \tilde{\mathcal{P}}_j, \quad (21.14)$$

which is unique because $\mathcal{V} \subset \mathcal{K}$ and $\tilde{\mathcal{P}}_j$ have no nonzero vector in common. Therefore \mathbf{P} has the unique decomposition

$$\mathbf{P} = \tilde{\mathbf{P}} + \mathbf{P}', \quad (21.15)$$

where

$$\mathbf{P}' = \mathbf{v} + \mathbf{K} = \mathbf{P} - \tilde{\mathbf{P}} \in \mathcal{P}_j \cap \mathcal{K} = \mathcal{P}'_j. \quad (21.16)$$

This decomposition and the fact that (21.7) implies $\tilde{\mathcal{P}}_j$ and $\mathcal{P}'_j \subset \mathcal{K}$ have no vector in common establishes (21.6).

So we deduce that

$$\begin{aligned} \mathcal{H} &= \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \tilde{\mathcal{P}}_1 \oplus \tilde{\mathcal{P}}_2 \oplus \cdots \oplus \tilde{\mathcal{P}}_n \oplus \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_n \\ &= \mathcal{U} \oplus \mathcal{V} \oplus \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_n, \end{aligned} \quad (21.17)$$

and since the \mathcal{P}'_j , $j = 1, 2, \dots, n$ are all contained in \mathcal{K} it follows that

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_n. \quad (21.18)$$

Now suppose that given $\mathbf{e} \in \mathcal{U}$ we can solve the equations

$$\mathbf{j} + \mathbf{J}_1 = \mathbf{L}(\mathbf{e} + \mathbf{E}_1), \quad \mathbf{J}_1 = -\mathbf{Y}\mathbf{E}_1, \quad \mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad \mathbf{E}_1, \mathbf{J}_1 \in \mathcal{V}, \quad (21.19)$$

where \mathbf{Y} is the \mathbf{Y} -operator associated with the subspace collection (21.18). From the Y -problem we have

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \in \mathcal{E}, \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \in \mathcal{J} \quad \mathbf{J}_2 = \mathbf{L}\mathbf{E}_2, \quad \mathbf{E}_2, \mathbf{J}_2 \in \mathcal{H}', \quad (21.20)$$

where

$$\mathcal{H}' = \mathcal{P}'_1 \oplus \mathcal{P}'_2 \oplus \cdots \oplus \mathcal{P}'_n. \quad (21.21)$$

Since

$$\mathbf{j} + \mathbf{J}_1 + \mathbf{J}_2 = \mathbf{L}(\mathbf{e} + \mathbf{E}_1 + \mathbf{E}_2), \quad (21.22)$$

we see that these fields solve the Z -problem

$$\mathbf{e}, \mathbf{j} \in \mathcal{U}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{j} + \mathbf{J} = \mathbf{L}(\mathbf{e} + \mathbf{E}), \quad (21.23)$$

and by definition $\mathbf{j} = \mathbf{Z}\mathbf{e}$. To solve (21.19) let Π_1 be the projection onto \mathcal{V} . Then (21.19) implies

$$-\mathbf{Y}\mathbf{E}_1 = \Pi_1\mathbf{L}(\mathbf{e} + \Pi_1\mathbf{E}_1), \quad (21.24)$$

giving

$$\mathbf{E}_1 = -\Pi_1(\mathbf{Y} + \Pi_1\mathbf{L}\Pi_1)^{-1}\Pi_1\mathbf{L}\mathbf{e}, \quad (21.25)$$

where the inverse is to be taken on the subspace \mathcal{V} . It follows that

$$\mathbf{j} + \mathbf{J}_1 = \mathbf{L}\mathbf{e} - \mathbf{L}\Pi_1(\mathbf{Y} + \Pi_1\mathbf{L}\Pi_1)^{-1}\Pi_1\mathbf{L}\mathbf{e}, \quad (21.26)$$

implying

$$\mathbf{Z} = \Gamma_0\mathbf{L}\Gamma_0 - \Gamma_0\mathbf{L}\Pi_1(\mathbf{Y} + \Pi_1\mathbf{L}\Pi_1)^{-1}\Pi_1\mathbf{L}\Gamma_0. \quad (21.27)$$

This formula is analogous to that given in (29.12) of Milton (2002).

To obtain a more explicit way of writing (21.27) let us suppose we are given a basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ of \mathcal{U} . Since (21.8) only has the trivial solution $\mathbf{u} = 0$ each space $\widetilde{\mathcal{P}}_j$ has dimension m . It then follows from (21.3) that \mathcal{V} has dimension $m(n-1)$. Also, for $i = 1, 2, \dots, n-1$, (21.3) implies $\Lambda_i\mathbf{u}_j$ has the unique decomposition

$$\Lambda_i\mathbf{u}_j = \sum_k w_{ijk}\mathbf{u}_k + \mathbf{v}_{ij}, \quad \mathbf{v}_{ij} \in \mathcal{V}, \quad (21.28)$$

for some set of constants w_{ijk} . To show that the vectors \mathbf{v}_{ij} , which number $m(n-1)$, are independent, let us suppose

$$0 = \sum_{i=1}^{n-1} \sum_{j=1}^m c_{ij}\mathbf{v}_{ij} = \sum_{i=1}^{n-1} \sum_{j=1}^m c_{ij}(\Lambda_i\mathbf{u}_j - \sum_{k=1}^m w_{ijk}\mathbf{u}_k). \quad (21.29)$$

By letting Λ_n act on this equation and taking into account that (21.8) only has the trivial solution $\mathbf{u} = 0$ we see that

$$\sum_{i=1}^{n-1} \sum_{j=1}^m \sum_{k=1}^m c_{ij}w_{ijk}\mathbf{u}_k = 0. \quad (21.30)$$

Then substituting this in (21.29) and letting Λ_i , $i \neq n$, act on (21.29) and again taking into account that (21.8) only has the trivial solution $\mathbf{u} = 0$ we obtain

$$\sum_{j=1}^m c_{ij}\mathbf{u}_j = 0, \quad (21.31)$$

which shows that all the c_{ij} must be zero. Therefore let us take the vectors \mathbf{v}_{ij} as our basis for \mathcal{V} .

The identities

$$\Pi_1\Lambda_i\Gamma_0\mathbf{u}_j = \mathbf{v}_{ij}, \quad \Gamma_0\Lambda_i\Gamma_0\mathbf{u}_j = \sum_k w_{ijk}\mathbf{u}_k, \quad (21.32)$$

which follow from (21.28) then gives the matrix representations for $\Pi_1 \Lambda_i \Gamma_0$ and $\Gamma_0 \Lambda_i \Gamma_0$ in these bases, when $i \neq n$. Using the fact that $\Lambda_n = \mathbf{I} - \sum_{i \neq n} \Lambda_i$ we obtain

$$\Gamma_0 \mathbf{L} \Gamma_0 = z_n \Gamma_0 + \sum_{i=1}^{n-1} (z_i - z_n) \Gamma_0 \Lambda_i \Gamma_0, \quad \Pi_1 \mathbf{L} \Gamma_0 = \sum_{i=1}^{n-1} (z_i - z_n) \Pi_1 \Lambda_i \Gamma_0. \quad (21.33)$$

Now for $p \neq n$ (and $i \neq n$) (21.28) implies (no sum over p)

$$\begin{aligned} \Lambda_p \mathbf{v}_{ij} &= \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) \Lambda_p \mathbf{u}_k \\ &= \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) (\mathbf{v}_{pk} + \sum_q w_{pkq} \mathbf{u}_q). \end{aligned} \quad (21.34)$$

Thus we deduce

$$\begin{aligned} \Gamma_0 \Lambda_p \Pi_1 \mathbf{v}_{ij} &= \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) \sum_q w_{pkq} \mathbf{u}_q, \\ \Pi_1 \Lambda_p \Pi_1 \mathbf{v}_{ij} &= \sum_k (\delta_{pi} \delta_{kj} - w_{ijk}) \mathbf{v}_{pk}, \end{aligned} \quad (21.35)$$

which gives the matrix representation for the operators $\Gamma_0 \Lambda_p \Pi_1$ and $\Pi_1 \Lambda_p \Pi_1$ in these bases ($p \neq n$), in terms of which we obtain the representation for the operators

$$\Gamma_0 \mathbf{L} \Pi_1 = \sum_{p=1}^{n-1} (z_p - z_n) \Gamma_0 \Lambda_p \Pi_1, \quad \Pi_1 \mathbf{L} \Pi_1 = z_n \Pi_1 + \sum_{p=1}^{n-1} (z_p - z_n) \Pi_1 \Lambda_p \Pi_1. \quad (21.36)$$

Thus all the matrices representing the operators entering (21.27), aside from \mathbf{Y} , only depend on the parameters w_{ijk} and these parameters can be obtained from the representation in the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ of \mathbf{Z} when the differences $z_i - z_n$, $i = 1, 2, \dots, n-1$ are small. To first order in these differences, (21.27), (21.33), and (21.36) imply

$$\mathbf{Z} \mathbf{u}_j \approx z_n \mathbf{u}_j + \sum_{i=1}^{n-1} (z_i - z_n) \sum_k w_{ijk} \mathbf{u}_k. \quad (21.37)$$

Thus knowing this expansion one can recover all the parameters w_{ijk} .

22 “Continued fraction expansions” of subspace collections.

The idea to developing the continued fraction is that by a succession of reduction and normalization operations one obtains a series of recursion relations

$$\mathbf{Z} = \Gamma_0 \mathbf{L} \Gamma_0 - \Gamma_0 \mathbf{L} \Pi_1 (\mathbf{Y} + \Pi_1 \mathbf{L} \Pi_1)^{-1} \Pi_1 \mathbf{L} \Gamma_0, \quad (22.38)$$

$$\mathbf{Y} = \mathbf{M}^{(1)} \mathbf{Z}^{(1)} \mathbf{K}^{(1)}, \quad (22.39)$$

$$\mathbf{Z}^{(1)} = \Gamma_0^{(1)} \mathbf{L}^{(1)} \Gamma_0^{(1)} - \Gamma_0^{(1)} \mathbf{L}^{(1)} \Pi_1^{(1)} (\mathbf{Y}^{(1)} + \Pi_1^{(1)} \mathbf{L}^{(1)} \Pi_1^{(1)})^{-1} \Pi_1^{(1)} \mathbf{L}^{(1)} \Gamma_0^{(1)}, \quad (22.40)$$

$$\mathbf{Y}^{(1)} = \mathbf{M}^{(2)} \mathbf{Z}^{(2)} \mathbf{K}^{(2)}, \quad (22.41)$$

$$\mathbf{Z}^{(2)} = \Gamma_0^{(2)} \mathbf{L}^{(2)} \Gamma_0^{(2)} - \Gamma_0^{(2)} \mathbf{L}^{(2)} \Pi_1^{(2)} (\mathbf{Y}^{(2)} + \Pi_1^{(2)} \mathbf{L}^{(2)} \Pi_1^{(2)})^{-1} \Pi_1^{(2)} \mathbf{L}^{(2)} \Gamma_0^{(2)}, \quad (22.42)$$

and so forth, until the dimension of the remaining space goes to zero, or until one (or more) of the assumptions necessary to proceed with the normalization or reduction operation does not hold. By substituting (22.39) in (22.38), then substituting (22.40) in the resulting expression, and subsequently substituting (22.41) in this expression, and so on, one develops the continued fraction expansion for \mathbf{Z} incorporating the variables z_1, z_2, \dots, z_n and, as one goes down the continued fraction, information contained in the series expansion (15.1) at successively higher and higher levels of truncation. We do not address in this book whether one can go ahead with the continued fraction expansion (and if so how) when the assumptions made to proceed with the normalization or reduction operation do not hold. In the process of developing the continued fraction through reduction and normalization operations, one could at those steps where one is dealing with a Y -problem make any desired reference transformation as described in Section 12. In this way one incorporates information at the subspace collection level that corresponds at the function level to known values of the function, and derivatives, at various points.

Such continued fraction expansions form the basis of the field equation recursion method for bounding the effective moduli of composites (Milton and Golden 1985; Milton 1987a, 1987b, 1991; Clark and Milton 1994; Clark 1997 and Chapter 29 of (Milton 2002) in the abstract theory of composites as described in Chapter 2 of this book (Milton 2016): see also Section 9.10 and Chapter 10 of Milton (2016)). The basic idea, at least when we have an orthogonal subspace collection, is that crude estimates or bounds on the operator $\mathbf{Z}^{(j)}$ or $\mathbf{Y}^{(j)}$ at some intermediate level j give through the above recursion relations good approximations or tight bounds on \mathbf{Z} or \mathbf{Y} incorporating the parameters that enter the recursion relations at the different levels up to level j (obtained from series expansions up to a given order of the solutions of the Z -problem or Y -problem).

Acknowledgments

G.W. Milton thanks his husband John K. Patton for suggesting the name superfunction.

References

- Baker, Jr., G. A. 1969, May. Best error bounds for Padé approximants to convergent series of Stieltjes. *Journal of Mathematical Physics* 10(5): 814–820. CODEN JMAPAQ. ISSN 0022-2488 (print), 1089-7658 (electronic), 1527-2427. (W. J. Thron) **MR 41 #3722** 30.25. URL http://jmp.aip.org/resource/1/jmapaq/v10/i5/p814_s1
- Baker, Jr., G. A. and P. R. Graves-Morris 1981. *Padé Approximants: Basic Theory. Part I. Extensions and Applications. Part II.* Reading, Massachusetts: Addison-Wesley. xviii + 215 pp. With a foreword by Peter A. Carruthers. ISBN 0-201-13512-4 (part 1), 0-521-30233-1 (part 1), 0-201-13513-2 (part 2), 0-521-30234-X (part 2). LCCN QC20.7.P3 .B35 PT.1-2 (1981). (Claude Brezinski) **MR 635620 (83a:41009b)** 41A15 (65Dxx 81E99 81F99).
- Bergman, D. J. 1978, July. The dielectric constant of a composite material — A problem in classical physics. *Physics Reports* 43(9):377–407.

- CODEN PRPLCM. ISSN 0370-1573 (print), 1873-6270 (electronic). URL <http://www.sciencedirect.com/science/article/pii/0370157378900091>
- Bergman, D. J. 1986. The effective dielectric coefficient of a composite medium: Rigorous bounds from analytic properties. In J. L. Ericksen, D. Kinderlehrer, R. V. Kohn, and J.-L. Lions (eds.), *Homogenization and Effective Moduli of Materials and Media*, pp. 27–51. Berlin / Heidelberg / London / etc.: Springer-Verlag. ISBN 0-387-96306-5. LCCN QA808.2 .H661 1986.
- Clark, K. E. 1997, September. A continued fraction representation for the effective conductivity of a two-dimensional polycrystal. *Journal of Mathematical Physics* 38 (9):4528–4541. CODEN JMAPAQ. ISSN 0022-2488 (print), 1089-7658 (electronic), 1527-2427. **MR 98f:82131** 82D25.
- Clark, K. E. and G. W. Milton 1994. Modeling the effective conductivity function of an arbitrary two-dimensional polycrystal using sequential laminates. *Proceedings of the Royal Society of Edinburgh* 124A(4):757–783. CODEN PRSEAE. ISSN 0080-4541.
- Dell’Antonio, G. F., R. Figari, and E. Orlandi 1986. An approach through orthogonal projections to the study of inhomogeneous or random media with linear response. *Annales de l’institut Henri Poincaré (A) Physique théorique* 44(1):1–28. CODEN AHPAAO. ISSN 0020-2339 (print), 2400-4863 (electronic). URL <http://eudml.org/doc/76310>
- Fokin, A. G. 1982, May 1. Iteration method in the theory of nonhomogeneous dielectrics. *Physica Status Solidi. B, Basic Research* 111(1):281–288. CODEN PSSBBD. ISSN 0370-1972 (print), 1521-3951 (electronic). URL <http://onlinelibrary.wiley.com/doi/10.1002/pssb.2221110131/abstract>
- Golden, K. M. and G. C. Papanicolaou 1983. Bounds for effective parameters of heterogeneous media by analytic continuation. *Communications in Mathematical Physics* 90(4):473–491. CODEN CMPHAY. ISSN 0010-3616. **MR 84k:78006** 78A25 (28A99 58E99).
- Grabovsky, Y. 1998. Exact relations for effective tensors of polycrystals. I: Necessary conditions. *Archive for Rational Mechanics and Analysis* 143(4):309–329. CODEN AVRMAW. ISSN 0003-9527 (print), 1432-0673 (electronic). (Robert Lipton) **MR 1657099 (2000c:74082)** 74Q15 (74A40 74E30 74F15).
- Grabovsky, Y. 2004. Algebra, geometry and computations of exact relations for effective moduli of composites. In G. Capriz and P. M. Mariano (eds.), *Advances in Multifield Theories of Continua with Substructure*, Modelling and Simulation in Science, Engineering and Technology, pp. 167–197. Boston, MA: Birkhäuser Verlag. ISBN 0-8176-4324-9. LCCN QA808.2 .A385 2004. **MR 2035115 (2004h:74002)** 74-06 (74Axx).
- Grabovsky, Y. and G. W. Milton 1998. Exact relations for composites: Towards a complete solution. *Documenta Mathematica, Journal der Deutschen Mathematiker-Vereinigung* Extra Volume ICM III:623–632. ISSN 1431-0635 (print), 1431-0643 (electronic). URL <http://www.emis.ams.org/journals/DMJDMV/xvol-icm/16/Milton.MAN.html>

- Grabovsky, Y., G. W. Milton, and D. S. Sage 2000, March. Exact relations for effective tensors of composites: Necessary conditions and sufficient conditions. *Communications on Pure and Applied Mathematics (New York)* 53(3):300–353. CODEN CPAMAT, CPMAMV. ISSN 0010-3640. URL <http://doi.org/d8k4vw>
- Grabovsky, Y. and D. S. Sage 1998. Exact relations for effective tensors of polycrystals. II: Applications to elasticity and piezoelectricity. *Archive for Rational Mechanics and Analysis* 143(4):331–356. CODEN AVRMAW. ISSN 0003-9527 (print), 1432-0673 (electronic). (Robert Lipton) **MR 1657103 (2000c:74083)** 74Q15 (74A40 74E30 74F15).
- Kantor, Y. and D. J. Bergman 1984. Improved rigorous bounds on the effective elastic moduli of a composite material. *Journal of the Mechanics and Physics of Solids* 32:41–62. CODEN JMPSA8. ISSN 0022-5096 (print), 1873-4782 (electronic).
- Kohler, W. and G. C. Papanicolaou 1982. Bounds for the effective conductivity of random media. In R. Burridge, S. Childress, and G. C. Papanicolaou (eds.), *Macroscopic Properties of Disordered Media: Proceedings of a Conference Held at the Courant Institute, June 1–3, 1981*, pp. 111–130. Berlin / Heidelberg / London / etc.: Springer-Verlag. ISBN 0-8224-8461-7, 0-273-08461-5. **MR 674963 (84m:82064)** 82A42 (49A29 82A70).
- Kröner, E. 1977, April. Bounds for the effective elastic moduli of disordered materials. *Journal of the Mechanics and Physics of Solids* 25(2):137–155. CODEN JMPSA8. ISSN 0022-5096 (print), 1873-4782 (electronic). URL <http://www.sciencedirect.com/science/article/pii/0022509677900096>
- Milton, G. W. 1979. Theoretical studies of the transport properties of inhomogeneous media. Unpublished report TP/79/1, University of Sydney, Sydney, Australia. 1–65 pp.
- Milton, G. W. 1981a, August 1. Bounds on the complex permittivity of a two-component composite material. *Journal of Applied Physics* 52(8):5286–5293. CODEN JAPIAU. ISSN 0021-8979 (print), 1089-7550 (electronic), 1520-8850. URL <http://scitation.aip.org/content/aip/journal/apl/37/3/10.1063/1.91895>
- Milton, G. W. 1981b, August 1. Bounds on the transport and optical properties of a two-component composite material. *Journal of Applied Physics* 52(8):5294–5304. CODEN JAPIAU. ISSN 0021-8979 (print), 1089-7550 (electronic), 1520-8850. URL <http://scitation.aip.org/content/aip/journal/jap/52/8/10.1063/1.329386>
- Milton, G. W. 1986. Modeling the properties of composites by laminates. In J. L. Ericksen, D. Kinderlehrer, R. V. Kohn, and J.-L. Lions (eds.), *Homogenization and Effective Moduli of Materials and Media*, pp. 150–174. Berlin / Heidelberg / London / etc.: Springer-Verlag. ISBN 0-387-96306-5. LCCN QA808.2 .H661 1986. **MR 859409 (87i:73006)** 73-02 (76-02 78-02).
- Milton, G. W. 1987a. Multicomponent composites, electrical networks and new types of continued fraction. I. *Communications in Mathematical Physics* 111(2):281–327. CODEN CMPHAY. ISSN 0010-3616. (V. Mastrangelo) **MR 89b:82084** 82A55 (73B99). URL <http://projecteuclid.org/euclid.cmp/1104159541>

- Milton, G. W. 1987b. Multicomponent composites, electrical networks and new types of continued fraction. II. *Communications in Mathematical Physics* 111(3):329–372. CODEN CMPHAY. ISSN 0010-3616. (V. Mastrangelo) **MR 89b:82085** 82A55 (73B99 73F99 94C05). URL <http://projecteuclid.org/euclid.cmp/1104159635>
- Milton, G. W. 1990. On characterizing the set of possible effective tensors of composites: The variational method and the translation method. *Communications on Pure and Applied Mathematics (New York)* 43(1):63–125. CODEN CPAMAT, CP-MAMV. ISSN 0010-3640. (John M. Ball) **MR 91c:73006** 73B27 (49S05 73K20).
- Milton, G. W. 1991. The field equation recursion method. In G. Dal Maso and G. F. Dell’Antonio (eds.), *Composite Media and Homogenization Theory: Proceedings of the Workshop on Composite Media and Homogenization Theory Held in Trieste, Italy, from January 15 to 26, 1990*, pp. 223–245. Basel, Switzerland: Birkhäuser Verlag. ISBN 0-8176-3511-4, 3-7643-3511-4. LCCN QA808.2 .C665 1991. **MR 1145740 (92h:73002)** 73-06 (00B25 49J45 73B27 73K20 76S05).
- Milton, G. W. 2002. *The Theory of Composites*. Cambridge, UK: Cambridge University Press. xxviii + 719 pp. Series editors: P. G. Ciarlet, A. Iserles, Robert V. Kohn, and M. H. Wright. ISBN 0-521-78125-6. LCCN TA418.9.C6 M58 2001.
- Milton, G. W. (ed.) 2016. *Extending the Theory of Composites to Other Areas of Science*. To appear.
- Milton, G. W. and K. M. Golden 1985. Thermal conduction in composites. In T. Ashworth and D. R. Smith (eds.), *Thermal Conductivity*, pp. 571–582. New York / London: Plenum Press. ISBN 0-306-41918-1. LCCN QC 320.8 I58 1983.
- Nicorovici, N. A., R. C. McPhedran, and G. W. Milton 1993, September 8. Transport properties of a three-phase composite material: The square array of coated cylinders. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 442(1916):599–620. CODEN PRLAAZ. ISSN 0080-4630.
- Papanicolaou, G. C. and S. R. S. Varadhan 1982. Boundary value problems with rapidly oscillating random coefficients. *Colloquia Mathematica Societatis János Bolyai* 27:835–873. ISSN 0139-3383. URL http://math.stanford.edu/~papanico/pubftp/pubs.old/pap_vara_79.pdf